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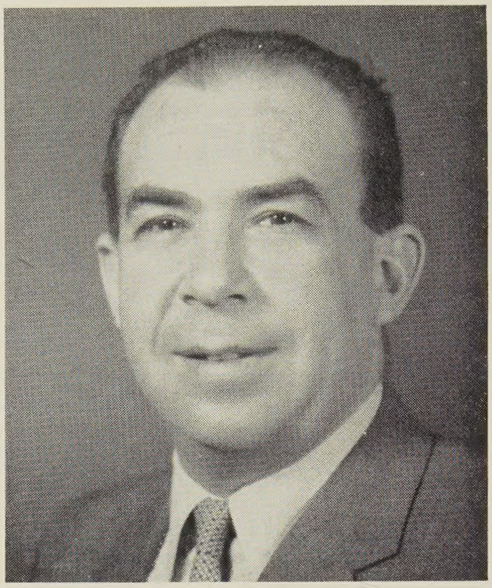

Robert M. Fano

Robert M. Fano was born on November 11, 1917 in Torino, Italy. He did part of his undergraduate work at the School of Engineering of Torino, Italy, before immigrating to this country in 1939. He received the degrees of B.S. and Sc.D. in electrical engineering from the Massachusetts Institute of Technology in 1941 and 1947, respectively.

For a time during World War II he was on the staff of the Radiation Laboratory of the Massachusetts Institute of Technology, where he worked on microwave components and filters. Except for this period he has been on the teaching staff of the Electrical Engineering Department at M.I.T. since 1941. He is presently Professor of Electrical Communications and a member of the staff of the Research Laboratory of Electronics. In addition to his teaching duties he was the Group Leader of the Radar Techniques Group of Lincoln Laboratory from 1950 to 1953. Dr. Fano has worked in the fields of network theory, electromagnetic theory, and information theory. He is co-author of Volume 9 of the Radiation Laboratory Series, and he has just completed the manuscript of a book on electromagnetic theory in collaboration with Dr. L. J. Chu.

He is a Fellow of the Institute of Radio Engineers, a Fellow of the American Academy of Arts and Science, and a member of Sigma Xi, Eta Kappa Nu, and other professional societies. He has been a member of the Administrative Committee of the PGIT ever since its formation and is presently Chairman of the Editorial Board of the TRANSACTIONS of the PGIT.

The Challenge of Digital Communication

ROBERT M. FANO

The necessity for transmitting digital information with ry low probability of error is becoming a major comunication problem. As is often the case in technical velopments, this problem is arising just at the time hen the basic theoretical tools for solving it are beginning take shape. A new challenge for communication gineers is being born.

The development of high-speed digital computers is aking possible the analysis and processing in real time increasingly large amounts of data. With their help, e detailed coordination of complex activities at widely parated locations is becoming technically feasible. ir-traffic control is an example of this. Thus, we can pect that in a relatively few years a major part of mmunication traffic will consist of information generated , or to be processed by, digital computers. By contrast, the past, practically all communication traffic was nerated by and directed to humans. A review of the lequacy of our communication techniques for such a dically different task is in order.

Communication between humans is greatly aided by e use of natural encoding schemes such as spoken nguage, developed by evolution to combat the various sturbances in the normal human environment. Above all, mmunication between humans hinges on the decoding peration performed by the brain of the receiver, which n recognize the transmitted message with a high egree of reliability even in the presence of substantial, npredictable disturbances. Thus, when a human is the ceiver of messages transmitted through an electric mmunication system, it is sufficient to maintain the vel of the electric disturbances within the rather broad mits for which the human coding and decoding system n operate satisfactorily.

Since present-day electric communication systems are rimarily designed for direct communication between umans they cannot be expected, in general, to perform tisfactorily as links between digital computers without rminal equipment that is able to perform encoding and ecoding operations similar to those performed by the uman brain. The analogy can be pushed further by isualizing these operations as being performed directly y the same digital computers that generate and use the formation.

Information theory throws considerable light on how igital information can be transmitted with low probbility of error. The probability of error per digit depends pon three parameters: the capacity of the communication hannel, the rate of transmission of information, and the ze of the "packages" of information that are encoded r transmission as single units. Shannon has shown that e probability of error after decoding can be reduced as much as desired, for fixed channel capacity and fixed rate of transmission, provided only that the former is greater than the latter. This result is accomplished by increasing the size of the packages of information that are encoded as single units. For the purpose of the present discussion, each package of information can be thought of as being encoded into a sequence of pulses (binary or n-ary). Some of them are information pulses that represent directly the digits that are to be transmitted; the rest are "check pulses" uniquely specified by the information pulses. The ratio of the two types of pulses remains constant when the size of the package is increased.

Feinstein, Elias, and Shannon have shown that by proper encoding, that is, by proper selection of the rules that specify the check pulses, the probability of error per digit can be made to decrease exponentially with the number of digits in each package. The coefficient in this exponent, by which the number of digits is multiplied, increases with the ratio of the channel capacity to the rate of transmission.

Thus, the probability of error can be decreased either by increasing the channel capacity in relation to the rate of transmission, or by increasing the number of digits in each package.

The main technical difficulty that must be solved before large packages of information can be encoded as units concerns the decoding apparatus. If we attempt to decode by following the most obvious procedure, the complexity of the decoding operation *grows* exponentially with the size of the information packages at the same time that the probability of error *decreases* exponentially. Thus, for example, the complexity would have to be doubled in order to reduce the probability of error by a factor of two. This would hardly be an attractive exchange. On the other hand, some recent work by Wozencraft indicates that with more efficient decoding procedures, the complexity of the decoding operation can be caused to increase at a rate that is slower than the square of the size of the information packages. Although many questions remain to be answered, it appears that the technical difficulties involved in the decoding operation are within the manageable range.

In the past, low probabilities of error have been achieved by using a channel capacity much larger than the transmission rate, with packages of minimum size. Ratios as great as ten are often used in binary pulse transmission. Great saving of channel capacity could be achieved in such cases by increasing the information packages. Furthermore, there are situations in which the channel capacity that can be made available is insufficient to provide an acceptable probability of error for the required rate of transmission.

While it is now technically simpler to use packages of minimum size and reduce the probability of error by increasing the channel capacity, consideration of long-range economic factors points to the other solution. The price of channel capacity is not likely to decrease very much; it is more likely to increase. In fact, the installation of new communication facilities involves a great deal of work, such as digging trenches for cables and erecting large antennas, which cannot be decreased much further by mass production. On the other hand, the equipment necessary for coding and decoding, which consist mostly of logical circuitry of the digital type, can be readily mass-produced once the design problems have been solved. Furthermore, the coding and decoding operations can be performed by general purpose digital computers. As suggested above, they may require only an increase in the size or speed of the computers that generate and receive the information. It seems clear that the cost of coding and decoding equipment is likely to decrease substantially in the future, while the cost of channel capacity is likely to remain of the same order of magnitude. While it is difficult to predict exactly where and when the two cost curves will cross, it is clear that they will do so in the not too distant future.

The key problem in achieving efficient transmission of information with very low probability of error lies in the decoding operation, which is in effect a form of information processing. This is one further piece of evidence that the fields of information processing and information transmission are moving steadily closer together, and are beginning to interact strongly with each other. Very likely we shall see them merge in the next few years into a broader field that will challenge the combined ranks of computer and communication engineers.

Some Comments on the Detection of Gaussian Signals in Gaussian Noise*

D. SLEPIAN†

Summary—It is pointed out that a frequently used mathematical iodel of the detection problem yields detection with arbitrarily mall probability of error in many cases of engineering interest. ome comments are made as to why the model is inadequate from n engineering standpoint.

I. The Comments

THE problem of detecting a Gaussian signal in Gaussian noise has been discussed by a number of authors during the past decade. A recent paper y Middleton[1] on this subject contains references to uch of the earlier work. Here we comment on several spects of this problem generally overlooked in the past.

The problem treated can be stated as follows. An bserver has available to him a sample of finite duration, (t), $0 \leq t \leq T$, of a stationary Gaussian process. It is nown to him that $x(t)$ is either a sample from the Gaussian nsemble A with mean zero and power-density spectrum $_A(f)$ or a sample from the Gaussian ensemble B with iean zero and power-density spectrum $\varphi_B(f)$. The observer ; to decide whether $x(t)$, $0 \leq t \leq T$ came from A or B. n most engineering applications, A is interpreted as gnal plus noise and B as noise alone.

The observer's decision as to which ensemble $x(t)$ ame from can be in error in two different ways: he can ssert that $x(t)$ came from A when indeed it came from ; or he can assert that $x(t)$ came from B when indeed came from A. We denote the probabilities of these two ypes of error by p_A and p_B respectively.

The main result of this paper is that for the spectra enerally considered in engineering problems, it is possible r the observer to make his decision with vanishingly nall probability of error of either kind. More specifically, e have the following

'heorem

Let $x(t)$, $0 \leq t \leq T$, be a sample from a stationary aussian ensemble with mean zero and power-density pectrum $\varphi_A(f)$ or $\varphi_B(f)$. If

$\varphi_A(f) \not\equiv \varphi_B(f)$,

$\varphi_A(f)$ is either bandlimited or rational in f,

$\varphi_B(f)$ is either bandlimited or rational in f, and if, in the ise where φ_A and φ_B are both rational

$$\lim_{f\to\infty} \frac{\varphi_A(f)}{\varphi_B(f)} \neq 1,$$

then there exist tests based on $x(t)$, $0 \leq t \leq T$, such that $p_A < \epsilon$ and $p_B < \epsilon$ for any preassigned $\epsilon > 0$. This result holds for arbitrarily small $T > 0$. It is to be noted that when A is signal plus noise and B is noise alone, the exceptional case in which φ_A and φ_B are both rational and $\lim_{f\to\infty} \varphi_A/\varphi_B = 1$ can occur only if the signal spectrum falls off more rapidly with large f than the noise spectrum. This case, for which the theorem makes no statement, is of some engineering interest. Examples are given by Middleton.[1]

The apparent contradiction of engineering intuition and experience implied by this theorem does not arise because of some trick hidden in the mathematical problem posed, but rather because the mathematical problem posed is not a good model of the detection problem encountered by the engineer. The model fails in *at least* two major respects even when the processes involved are truly Gaussian and stationary. In the first place, in practice the spectra φ_A and φ_B are not known exactly to the observer. The power spectra "known" to him are the results of measurements (frequently crude) or of educated guesses. In the second place, the observer does not have available to him a sample of either process in the sense implied by the mathematical notation $x(t)$, $0 \leq t \leq T$. He cannot, for example, meaningfully differentiate his observed sample 3000 times, nor can he meaningfully measure the values of the observed sample at time instants separated by only 10^{-100} seconds.

It will be seen below that the tests by which the results of the theorem can be obtained require the observer to make use of arbitrarily fine detail of the observed sample and required him to know precisely the behavior of φ_A and φ_B as $f \to \infty$. If it is assumed that φ_A and φ_B are known precisely to the observer, but that some restriction is placed upon his ability to measure the sample $x(t)$, the resulting model of the engineering detection problem still retains its unrealistic features. For example, if the observer has available to him only the quantities $x(jT/n) + \epsilon_j$, $j = 0, 1, \cdots, n$, where the measurement errors ϵ_j are assumed to be normal quantities of known mean and covariance, then in many cases the probabilities of error, p_A and p_B, can still be made as small as desired by making n large. To make a model of the detection problem satisfactory for engineering use it seems necessary to assume *both* 1) incomplete knowledge by the observer of

* Manuscript received by the PGIT, November 15, 1957. he work reported here was supported in part by the Office of aval Research under Contract Nonr 210 (00).

† Bell Telephone Labs., Inc., Murray Hill, N. J.

[1] D. Middleton. "On the detection of stochastic signals in lditive normal noise—Part I," IRE Trans. on Information heory, vol. IT-3, pp. 86–121; June, 1957.

the spectra φ_A and φ_B and 2) some restrictions on the observer's ability to measure fine detail of the actual process made available to him. We cannot, however, enter further into these matters here.

It is worth pointing out that in some published papers on this detection problem, "best" solutions are obtained in the case where the observer has available to him the quantities $x(jT/n)$, $j = 0, 1, \cdots, n$. In a manner which must be regarded as purely heuristic, limiting forms are written down for $n \to \infty$ and are presented as solutions to the problem for the case in which the observer measures $x(t)$ continuously in $0 \leq t \leq T$. In view of the foregoing theorem, these "continuous solutions" must be treated with some caution.

II. Proof of Theorem

The proof of the theorem follows immediately from results available in the literature. If φ_A and φ_B are band-limited, then with probability one the sample functions of A and B are analytic.[2] The sample functions can, therefore, be expanded in a Taylor series of radius of convergence $\rho > 0$ about any point. Since the process is stationary, ρ is independent of the point about which the expansion is made. The observer can, therefore, obtain the values of $x(t)$ everywhere from its values in $0 \leq t \leq T$ by making repeated Taylor series expansions. The autocorrelation of $x(t)$ can then be computed and the Fourier transform of this autocorrelation obtained. If $x(t)$ came from A, this transform will be φ_A with probability one; if $x(t)$ came from B, this transform will be φ_B with probability one.[3]

In the case of rational spectra, we make use of a theorem due to Baxter.[4] As adapted for our purposes, this theorem can be stated as follows. Let $x(t)$ be a sample from a stationary Gaussian process with mean zero and covariaance function $r(\tau) = E\, x(t)x(t + \tau)$. Let $r(\tau)$ be continuous and have a uniformly bounded second derivative for $\tau \neq 0$. Let

$$y_n = \sum_{j=0}^{n-1} \left[x\left\{\frac{(j+1)T}{n}\right\} - x\left\{\frac{jT}{n}\right\}\right]^2 \tag{1}$$

and $\alpha = -2Tr'(0+)$. Then $\lim_{n\to\infty} Pr\{|y_n - \alpha| > \epsilon\} = 0$ for every $\epsilon > 0$. That is, the test function (1) converges in probability to the jump in the first derivative of the covariance function at the origin.

To apply the theorem to the problem at hand, suppose that for large f, $\varphi_A \sim a/f^2$ and $\varphi_B \sim b/f^{2p}$, $p \geq 1$. Let $r_A(\tau)$ and $r_B(\tau)$ be the covariance functions of A and B. Then $-2\, r_A'(0+) = (2\pi)^2\, a$ and $-2r_B'(0+)$ is zero if $p > 1$ and is $(2\pi)^2 b$ if $p = 1$. The test function y_n of (1), then, will have a value close to $(2\pi)^2 aT$ with high probability when n is large and $x(t)$ did come from A. When $x(t)$ came from B, y_n will with high probability be close to zero if $p > 1$ and will be close to $(2\pi)^2 bT$ with high probability if $p = 1$. The rules for the observer can be given in the usual form of a threshold test. If $p > 1$ choose a threshold, $\bar{y}$, between zero and $(2\pi)^2 aT$. If $y_n \geq \bar{y}$, say that $x(t)$ came from A. If $y_n < \bar{y}$ say that $x(t)$ came from B. If $p = 1$, choose a threshold, $\bar{y}$, that lies between $(2\pi)^2 aT$ and $(2\pi)^2 bT$. If $a > b$, say that $x(t)$ came from A when $y_n \geq \bar{y}$ and say that $x(t)$ came from B when $y_n < \bar{y}$. If $b > a$, reverse the decision rule. The convergence of y_n in probability assures us that by taking n sufficiently large, both p_A and p_B can be made arbitrarily small by the above procedure unless possibly $p = 1$ and $a = b$.

To adapt Baxter's theorem to our problem when both $r_A(\tau)$ and $r_B(\tau)$ have a continuous first derivative at $\tau = 0$, we note[5] that if $\varphi_A \sim a/f^{2m}$, then $-2r^{(2m-1)}(0+) = (-1)^{m-1}(2\pi)^{2m}a$, *i.e.*, the $(2m - 1)$ *st* derivative of the covariance function has a jump at the origin of magnitude $(2\pi)^{2m}a$. Derivatives of order less than $2m - 1$ are continuous at $\tau = 0$. Furthermore, the process A is differentiable[6] $m-1$ times and $(-1)^m\, r^{(2m-2)}(\tau)$ is the covariance function of the derived process $x^{(m-1)}(\tau)$.

If, then, $\varphi_A \sim a/f^{2m}$ and $\varphi_B \sim b/f^{2(m+p)}$, $p \geq 0$, it is only necessary to differentiate $x(t)$ $m-1$ times before applying the test function (1) to obtain the same detection results as before. That is, the observer uses

$$y_n = \sum_{0}^{n-1} \left[z\left\{\frac{(j+1)T}{n}\right\} - z\left\{\frac{jT}{n}\right\}\right]^2,$$

where $z(t) = [d^{m-1}/dt^{m-1}]\, x(t)$, and decision rules, and thresholds analogous to those used in the case $m = 1$ already treated.

This same test will clearly distinguish between the two ensembles when φ_A is rational, $\varphi_A \sim a/f^{2m}$ say, and φ_B is band-limited.

III. Some Other Forms of the Test

As already noted, the detection problem is frequently restricted to allow the observer access to only a finite set of points, $x(jT/n)$, $j = 0, 1, \cdots, n$, or in the limit a denumerable set of points. The tests described in the preceding paragraph required, for the most part, using infinitely many values of $x(t)$ even for finite values of n, *e.g.*, derivatives were taken. It is of some interest, then, to show that detection with arbitrarily small values of p_A and p_B is possible using only a finite number of values of $x(t)$. We treat here only the case in which φ_A and φ_B are rational and the observer has access to $x(jT/n)$, $j = 0, 1, \cdots, n$. The results obtained are generalizations of Baxter's theorem.

[2] M. Loeve. "Probability Theory," D. Van Nostrand Co., Inc., New York, N. Y., corollary 3, p. 471; 1955.

[3] The reasoning here requires A and B to be ergodic. This condition is fulfilled for the processes under discussion. See U. Grenander, "Stochastic processes and statistical inference," *Arkiv. for Matematik*, band 1, no. 17, theorem, p. 257; 1950.

[4] Glen Baxter, "A strong limit theorem for Gaussian processes," *Proc. Amer. Math. Soc.*, vol. 7, pp. 522–527; June, 1956.

[5] See J. L. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y., pp. 542–551; 1953, for a discussion of the results stated in this paragraph.

[6] *Ibid.* A separable version of the process is assumed.

Let $\varphi_A \sim a/f^{2s}$ and $n = mq$ where s, m and q are positive integers and $s \geq m$. Consider

$$y_n = \left(\frac{n}{T}\right)^{2m-2} \sum_{j=0}^{q-1} \left(\sum_{k=0}^{m} \binom{m}{k} (-1)^k x\left\{\frac{(jm+k)T}{n}\right\}\right)^2. \quad (2)$$

The expected value of y_n is

$$Ey_n = \frac{n^{2m-1}}{mT^{2m-2}} \sum_{k=0}^{m} \sum_{l=0}^{m} \binom{m}{k}\binom{m}{l} (-1)^{k+l} r_A\left[\frac{(l-k)T}{n}\right].$$

On introducing the Fourier integral representation for r_A, this can be written

$$Ey_n = \frac{n^{2m-1}}{mT^{2m-2}} \int_{-\infty}^{\infty} df \varphi_A(f) \cdot \sum_{k=0}^{m} \sum_{l=0}^{m} \binom{m}{k}\binom{m}{l} (-1)^{k+l} e^{2\pi i(l-k)f(T/n)}$$

$$= \frac{2^{2m} n^{2m-1}}{mT^{2m-2}} \int_{-\infty}^{\infty} df \varphi_A(f) \sin^{2m} \frac{\pi T f}{n}$$

$$= \frac{2^{2m}\pi^{2m-1}T}{m} \int_{-\infty}^{\infty} d\xi \left(\frac{n\xi}{\pi T}\right)^{2m} \varphi_A\left(\frac{n\xi}{\pi T}\right)\left(\frac{\sin \xi}{\xi}\right)^{2m}.$$

It then follows easily from the asymptotic behavior of φ_A that if $s > m$

$$\lim_{q\to\infty} Ey_{qm} = 0$$

while if $s = m$,

$$\lim_{q\to\infty} Ey_{qm} = ac_m$$

where

$$c_m = \frac{2^{2m}\pi^{2m-1}T}{m} \int_{-\infty}^{\infty} d\xi \left(\frac{\sin \xi}{\xi}\right)^{2m}.$$

The variance of y_n can also be computed in a straightforward manner. One finds

$$\text{Var } y_n = \left(\frac{n}{T}\right)^{4m-4} \sum_{j,\sigma=0}^{q-1} \sum_{\substack{k,l \\ \mu,\nu=0}}^{m} \binom{m}{k}\binom{m}{l}\binom{m}{\mu}\binom{m}{\nu} (-1)^{k+l+\mu+\nu}$$

$$\cdot \left[r_A\left\{[(j-\sigma)m + k - \mu]\frac{T}{n}\right\} r_A\left\{[(j-\sigma)m + l - \nu]\frac{T}{n}\right\} + r_A\left\{[(j-\sigma)m + k - \nu]\frac{T}{n}\right\} r_A\left\{[(j-\sigma)m + l - \mu]\frac{T}{n}\right\}\right].$$

Introduce the Fourier integral representation of r_A, interchange summation and integration and perform the sums. There results

$$\text{Var } y_n = 2^{4m-1}\left(\frac{n}{T}\right)^{4m-4} \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \varphi_A(f)\varphi_A(f') \cdot \left[\frac{\sin \pi mq(f-f')\dfrac{T}{n}}{\sin \pi m(f-f')\dfrac{T}{n}}\right]^2 \left[\sin \pi f \frac{T}{n} \sin \pi f' \frac{T}{n}\right]^{2m}$$

or, after an appropriate change of variables

$$\text{Var } y_n = \frac{2^{4m-1}\pi^{2m-2}T^2}{n^2} \cdot \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \left(\frac{n\xi}{\pi T}\right)^{2m} \varphi_A\left(\frac{n\xi}{\pi T}\right)\left(\frac{n\eta}{\pi T}\right)^{2m} \varphi_A\left(\frac{n\eta}{\pi T}\right) \cdot \left[\frac{\sin \xi}{\xi}\frac{\sin \eta}{\eta}\right]^{2m}\left[\frac{\sin mq(\xi-\eta)}{\sin m(\xi-\eta)}\right]^2. \quad (3)$$

We now show that $\lim_{q\to\infty} \text{Var } y_{mq} = 0$. Since all factors in the integrand of (3) are nonnegative and since $f^{2m}\varphi_A(f)$ is bounded from above by our assumption, it follows that

$$\text{Var } y_{mq} < \frac{d}{q} h_q \quad (4)$$

where d does not depend on n or q and

$$h_q = \frac{1}{q} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \left[\frac{\sin \dfrac{\xi}{m}}{\dfrac{\xi}{m}} \frac{\sin \dfrac{\eta}{m}}{\dfrac{\eta}{m}}\right]^{2m} \left[\frac{\sin q(\xi-\eta)}{\sin (\xi-\eta)}\right]^2.$$

Now

$$h_q = \frac{1}{q} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_{j\pi}^{(j+1)\pi} d\xi_j \int_{k\pi}^{(k+1)\pi} d\eta_k \cdot \left[\frac{\sin \dfrac{\xi_j}{m}}{\dfrac{\xi_j}{m}} \frac{\sin \dfrac{\eta_k}{m}}{\dfrac{\eta_k}{m}}\right]^{2m} \left[\frac{\sin q(\xi_j-\eta_k)}{\sin (\xi_j-\eta_k)}\right]^2$$

$$= \frac{1}{q}\int_0^{\pi} d\xi \int_0^{\pi} d\eta \left[\frac{\sin q(\xi-\eta)}{\sin (\xi-\eta)}\right]^2 h(\xi)h(\eta)$$

$$= \frac{1}{q}\int_0^{\pi} dx \left[\frac{\sin qx}{\sin x}\right]^2 g(x) \quad (5)$$

where

$$h(\xi) = \sum_{j=-\infty}^{\infty} \left[\frac{\sin \dfrac{1}{m}(\xi + j\pi)}{\dfrac{1}{m}(\xi + j\pi)}\right]^{2m}$$

and

$$g(x) = \int_x^{2\pi - x} dy\, h[\tfrac{1}{2}(y+x)]h[\tfrac{1}{2}(y-x)].$$

The square bracket in (5) is the Fejer kernel studied in Fourier theory.[7] As $q \to \infty$, $h_q \to \pi\, g(0+)$ which is finite. From (4) it follows that $\text{Var } y_{mq} \to 0$ as $q \to \infty$.

The preceding paragraphs show that if $\varphi_A \sim a/f^{2s}$, then y_{mq} of (2) converges in probability to zero if $s > m$ and to ac_m if $s = m$. Consider now the problem of distinguishing between the stationary Gaussian ensembles A and B where $\varphi_A \sim a/f^{2m}$ and $\varphi_B \sim b/f^{2(m+p)}$, $p \geq 0$. Form the test function (2) from the observed samples

[7] E. T. Whittaker and G. N. Watson, "Modern Analysis," The MacMillan Co., New York, N. Y., pp. 170–171; 1947.

$x(jT/n)$, $j = 0, 1, \cdots, n$, $n = mq$. If $p > 0$, choose a threshold $\bar{y}$ between zero and $c_m a$. Use the decision rule "sample came from A if $y_{qm} \geq \bar{y}$; sample came from B if $y_{qm} < \bar{y}$." If $p = 0$ and $a > b$, choose a threshold $\bar{y}$ between $c_m a$ and $c_m b$. Use the decision rule "sample came from A if $y_{qm} \geq \bar{y}$; sample came from B if $y_{qm} < \bar{y}$." If $p = 0$ and $a < b$, reverse the decision rule. By choosing q, and hence n, sufficiently large p_A and p_B can be made arbitrarily small.

In the test just described, it was necessary to use arbitrarily many values of the observed sample $x(t)$ to make p_A and p_B arbitrarily small. It is interesting to note that if $\varphi_A \sim a/f^{2m}$ and $\varphi_B \sim b/f^{2(m+p)}$, $p > 0$, it is possible to obtain these same results using only the $m + 1$ observed values $x(jT/n)$, $j = 0, 1, \cdots, m$. The test function to be used in this case is

$$y_n = \left[\left(\frac{n}{T}\right)^m \sum_{j=0}^{m} \binom{m}{j}(-1)^j x\left(\frac{jT}{n}\right)\right]^2.$$

A threshold $\bar{y}$ is chosen. If $y_n \geq \bar{y}$, the observer says $x(t)$ came from A; otherwise that $x(t)$ came from B. It is easy to show that by making n large enough and by choosing $\bar{y}$ properly ($\bar{y}$ increases with n) p_A and p_B can be made arbitrarily small. We omit the details here. It is curious that the total observation time of the process needed to make this test becomes vanishingly small as the accuracy of the test improves.

We note that the results of this section are valid for a larger class of stationary Gaussian processes than those having rational power spectral densities. They are valid, for example, for all bounded φ_A and φ_B having the prescribed asymptotic behavior.

IV. Likelihood Ratio Tests

Let the quantities $x_j = x(jT/n)$, $j = 0, 1, \cdots, n$ be available to the observer. The x_j are jointly Gaussian with mean zero. If $x(t)$ comes from A, the covariance matrix of the x_j has elements $A_{ij} = r_A[(i - j)(T/n)]$; if the x_j came from B, their covariance matrix has elements $B_{ij} = r_B[(i - j)(T/n)]$. Let the corresponding Gaussian probability densities be denoted by $g_A(x_0, \cdots, x_n)$ and $g_B(x_0, \cdots, x_n)$. Many desirable tests[1] to distinguish between A and B have decision rules of the form: choose A if the likelihood ratio g_A/g_B exceeds a threshold value; choose B otherwise. For example, the Neyman-Pearson test which minimizes p_A for a given value of p_B is of this form. Since for Gaussian densities, the likelihood ratio depends monotonely on the quadratic form

$$y_n = \frac{1}{n} \sum_{0}^{n} [A_{ij}^{-1} - B_{ij}^{-1}] x_i x_j, \tag{6}$$

the characteristics of these threshold tests can be determined by studying the random variable y_n.

Let $f_{nA}(y_n)$ and $f_{nB}(y_n)$ denote the probability density function of (6) according as $x(t)$ came from A or B respectively. Then the probabilities of the two kinds of decision errors are

$$p_A = \int_{\bar{y}}^{\infty} f_{nB}(y)\, dy, \qquad p_B = \int_{-\infty}^{\bar{y}} f_{nA}(y)\, dy \tag{7}$$

where $\bar{y}$ is the threshold. To investigate the behavior of these errors for large n, it is necessary to study $f_{nB}(y)$ and $f_{nA}(y)$.

Such a study appears to be quite difficult for general covariance functions $r_A(\tau)$ and $r_B(\tau)$. The results of Section III show, however, that by suitable choice of $\bar{y}$, p_A and p_B can be made arbitrarily small in the case where r_A and r_B are derived from rational spectra not possessing identical asymptotic behavior. For, like (6), the tests considered in Section III use only the quantities x_j and they lead to arbitrarily small probabilities p_A and p_B. From the optimum properties of the Neyman-Pearson test, (6) must do as well or better for each n.

Much more can be said in detail about f_{nA} and f_{nB} in the case where r_A and r_B are rational, but we put this off for a possible later paper. We note here only that the exclusion of the case in which $\varphi_A/\varphi_B \rightarrow 1$ is not unnecessary in general, for if $r_A = \alpha_1 e^{-\beta_1|\tau|}$ and $r_B = \alpha_2 e^{-\beta_2|\tau|}$, p_A and p_B of (7) cannot simultaneously be made arbitrarily small when $\alpha_1\beta_1 = \alpha_2\beta_2$.

V. Conclusions

The purely mathematical problem of testing the hypothesis that $x(t)$, $0 \leq t \leq T$, came from the stationary Gaussian ensemble A with mean zero and known spectral density φ_A against the alternate hypothesis that $x(t)$ came from the stationary Gaussian ensemble B with mean zero and known spectral density φ_B poses many difficult and interesting questions. In particular, necessary and sufficient conditions that the test be made with arbitrarily small probability of error are not known.[8] For the class of rational and bandlimited spectra (with some minor exceptions), it *is* possible to decide with arbitrarily small probability of error which ensemble $x(t)$ was drawn from. This mathematical hypothesis-testing problem is, in general, a poor model of the detection problem faced by the engineer. An adequate mathematical model of the detection problem must not assume perfect *a priori* knowledge of φ_A and φ_B and must place limitations on the observer's ability to measure $x(t)$.

VI. Acknowledgment

The author is indebted to his colleagues S. P. Lloyd and H. O. Pollak for many helpful discussions and suggestions during the course of this work.

[8] The earliest statement known to the author of a sufficient condition is contained in Grenander, *op. cit.*, p. 221. There the case $r_A(\tau) = \sigma^2 r_B(\tau)$ is treated and it is shown that p_A and p_B can be made arbitrarily small if $\sigma^2 \neq 1$.

A Useful Theorem for Nonlinear Devices Having Gaussian Inputs*

ROBERT PRICE†

Summary—**If and only if the inputs to a set of nonlinear, zero-memory devices are variates drawn from a Gaussian random process, a useful general relationship may be found between certain input and output statistics of the set. This relationship equates partial derivatives of the (high-order) output correlation coefficient taken with respect to the input correlation coefficients, to the output correlation coefficient of a new set of nonlinear devices bearing a simple derivative relation to the original set. Application is made to the interesting special cases of conventional cross-correlation and autocorrelation functions, and Bussgang's theorem is easily proved. As examples, the output autocorrelation functions are simply obtained for a hard limiter, linear detector, clipper, and smooth limiter.**

IN THE COURSE of investigating the asymptotic frequency behavior of power spectra resulting from the passage of noise through zero-memory nonlinear devices, an interesting, unique property of Gaussian processes has been encountered, which does not appear to have been previously reported.

Statement of the Theorem

Assume $x_1, x_2, \cdots, x_n$ to be random variables from a Gaussian process whose nth order joint probability density is given by:[1]

$$p(x_1, x_2, \cdots, x_n) = (2\pi)^{-n/2} \mid M_n \mid^{-1/2} \cdot \exp\left\{-\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{M_{rs}}{\mid M_n \mid}(x_r - \overline{x_r})(x_s - \overline{x_s})\right\} \tag{1}$$

where $\mid M_n \mid$ is the determinant of $M_n = [\rho_{rs}]$ and $\rho_{rs} = \overline{x_r x_s} - \overline{x_r}\,\overline{x_s} = \rho_{sr}$ is the correlation coefficient of x_r and x_s. The means of x_r and x_s are $\overline{x_r}$ and $\overline{x_s}$, respectively. M_{rs} is the cofactor of ρ_{sr} in M_n.

Let there be n zero-memory nonlinear devices (linearity of course being included as a special case) specified by the input-output relationship $f_i(x)$, $i = 1, 2, \cdots, n$. Let each x_i be the single input to a corresponding $f_i(x)$, and designate the nth-order correlation coefficient of the outputs as:

$$R = \overline{\prod_{i=1}^{n} f_i(x_i)} \tag{2}$$

where the bar denotes the average taken over all x_i. Then, with weak restrictions on the $f_i(x)$, we have the following theorem for the partial derivatives of R with respect to the input correlation coefficients:

$$\frac{\partial^k R}{\prod_{m=1}^{N}(\partial\rho_{r_m s_m})^{k_m}} = \left(\frac{1}{2}\right)^{\sum_{m=1}^{N} k_m \delta_{r_m s_m}} \left[\overline{\prod_{i=1}^{n} f_i^{\left(\sum_{m=1}^{N}\epsilon_{im}k_m\right)}(x_i)}\right] \tag{3}$$

where r_m and s_m, $m = 1, 2, \cdots, N$, are integers lying between 1 and n, inclusive, and are not necessarily distinct. The k_m are positive integers, with $k = \sum_{m=1}^{N} k_m$. ϵ_{im} is the number of times i appears in (r_m, s_m). $\delta_{r_m s_m}$ is the Kronecker δ function, $\delta_{r_m s_m} = 1$ for $r_m = s_m$, 0 for $r_m \neq s_m$. The symbol $f_i^{(q)}(x_i)$ denotes the qth derivative of $f_i(x)$, taken at x_i.

Furthermore, not only is the above theorem true for inputs having an nth-order joint Gaussian distribution, but it holds true *only* for such inputs if the $f_i(x)$ are allowed to be of general form.

Proof

We now prove that in order for (3) to be satisfied it is both sufficient and necessary that the x_i have the joint probability density given by (1). Assume that each $f_i(x)$ can be represented by the sum of two Laplace transforms,[2]

$$f_i(x) = \frac{1}{2\pi j}\int_{C_{i+}} h_{i+}(u)e^{jux}\,du + \frac{1}{2\pi j}\int_{C_{i-}} h_{i-}(u)e^{jux}\,du \tag{4}$$

where

$$\left.\begin{aligned} h_{i+}(u) &= \int_0^{\infty} f_i(x)e^{-jux}\,dx \\ h_{i-}(u) &= \int_{-\infty}^{0} f_i(x)e^{-jux}\,dx \end{aligned}\right\} \tag{5}$$

and the C_{i+} and C_{i-} are appropriate contours. Without assuming any particular form for $p(x_1, x_2, \cdots, x_n)$ for the present,

$$R = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \cdot \prod_{i=1}^{n} f_i(x_i)p(x_1, x_2, \cdots, x_n)\,dx_1\,dx_2\cdots dx_n. \tag{6}$$

Substituting (4) in (6) and inverting the order of integration, following Rice's characteristic function method,[3]

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[1] H. Cramér, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J., sec. 24.2; 1946.

[2] D. V. Widder, "The Laplace Transform," Princeton University Press, Princeton, N. J., ch. 6; 1946.

[3] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, pp. 282–332, July, 1944; and vol. 24, pp. 46–156; January, 1945. See sec. 4.8.

$$R = \frac{1}{(2\pi j)^n} \sum{}' \int_{C_{1\pm}} \int_{C_{2\pm}} \cdots \int_{C_{n\pm}} \cdot \prod_{i=1}^{n} h_{i\pm}(u_i)\theta(u_1, u_2, \cdots, u_n)\, du_1\, du_2 \cdots du_n \quad (7)$$

where $\sum'$ denotes a summation over all possible $\pm$ combinations and $\theta(u_1, u_2, \cdots, u_n)$ is the nth-order characteristic function:

$$\theta(u_1, u_2, \cdots, u_n) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p(x_1, x_2, \cdots, x_n) \cdot \exp\left(j \sum_{i=1}^{n} u_i x_i\right) dx_1\, dx_2 \cdots dx_n \quad (8)$$

with $j = \sqrt{-1}$.

We find a necessary condition for (3) to be satisfied by setting $N = 1 = k = k_1$. The partial derivative of the left-hand side of (3) is taken on θ in the integrand of (7), and the derivatives of the right-hand side are taken using (4). Thus the necessary condition:

$$\sum{}' \int_{C_{1\pm}} \int_{C_{2\pm}} \cdots \int_{C_{n\pm}} \prod_{i=1}^{n} h_{i\pm}(u_i) \left\{ \frac{\partial \theta(u_1, u_2, \cdots, u_n)}{\partial \rho_{r_1 s_1}} + \left(\frac{1}{2}\right)^{\delta_{r_1 s_1}} u_{r_1} u_{s_1} \theta(u_1, u_2, \cdots, u_n) \right\} du_1\, du_2 \cdots du_n = 0 \quad (9)$$

is obtained. The term in braces must be zero in order to satisfy (9) for arbitrary $f_i(x)$ and hence $h_{i\pm}(u)$. Integrating the resulting equation for all (r_1, s_1) (but taking into account that $\rho_{rs} = \rho_{sr}$),

$$\log \theta(u_1, u_2, \cdots, u_n) = -\frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \rho_{rs} u_r u_s + g(u_1, u_2, \cdots, u_n) \quad (10)$$

where g is some function which must now be found.

Let $\rho_{rs} = 1$ for all (r, s). Then all the x_i are completely correlated, and $p(x_1, x_2, \cdots, x_n)$ can be written:

$$p(x_1, x_2, \cdots, x_n) = p(x_1) \prod_{i=2}^{n} \delta(x_i - x_1 + \overline{x_1} - \overline{x_i}) \quad (11)$$

where $\delta(x)$ is the Dirac δ function. Substituting (11) in (8), θ is of the form:

$$\theta(u_1, u_2, \cdots, u_n) = \exp\left(j \sum_{i=1}^{n} u_i \overline{x_i}\right) g_1\left(\sum_{i=1}^{n} u_i\right), \quad \text{for all } \rho_{rs} = 1 \quad (12)$$

where

$$g_1(u) = \int_{-\infty}^{+\infty} p_1(x_1 - \overline{x_1}) e^{ju(x_1 - \overline{x_1})}\, d(x_1 - \overline{x_1}). \quad (13)$$

Similarly, when $\rho_{11} = 1$ r, $\rho_{1r} = \rho_{r1} = -1$ for all $r \neq 1$, and $\rho_{rs} = 1$ for all r or $s \neq 1$, then $x_2, x_3, \cdots, x_n$ are completely correlated with $(-x_1)$ and we obtain:

$$\theta(u_1, u_2, \cdots, u_n) = \exp\left(j \sum_{i=1}^{n} u_i \overline{x_i}\right) g_1\left(u_1 - \sum_{i=2}^{n} u_i\right),$$
$$\text{for } \rho_{11} = 1,\ \rho_{1r} = \rho_{r1} = -1 \text{ for all } r \neq 1, \text{ and } \rho_{rs} = 1 \text{ for all } r, \quad s \neq 1. \quad (14)$$

Substituting (12) in (10), we find:

$$g(u_1, u_2, \cdots, u_n) = j \sum_{i=1}^{n} u_i \overline{x_i} + g_2\left(\sum_{i=1}^{n} u_i\right) \quad (15)$$

where $g_2(u) = \log g_1(u) + u^2/2$. On the other hand, substituting (14) in (10) yields

$$g(u_1, u_2, \cdots, u_n) = j \sum_{i=1}^{n} u_i \overline{x_i} + g_2\left(2u_1 - \sum_{i=1}^{n} u_i\right). \quad (16)$$

Since u_1 and $\sum_{i=1}^{n} u_i$ may be considered as independent variables, the only solution which renders (15) and (16) compatible is $g_2(u) = K$, a constant. Thus, finally, we have from (10) and (15) the necessary condition:

$$\theta(u_1, u_2, \cdots, u_n) = \exp\left[-\frac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} \rho_{rs} u_r u_s + j \sum_{i=1}^{n} u_i \overline{x_i} + K\right]. \quad (17)$$

This is recognized to be the characteristic function of the n-dimensional Gaussian distribution[4] of (1) ($K = 0$ for proper normalization).

It is now a simple matter to prove the sufficiency of (17), and hence (1), for satisfying (3). Using (17) in (7), and remembering that $\rho_{rs} = \rho_{sr}$,

$$\frac{(-1)^k\, \partial k_R}{\prod_{m=1}^{N} (\partial \rho_{r_m s_m})^{k_m}} = \left(\frac{1}{2}\right)^{\sum_{m=1}^{N} k_m \delta_{r_m s_m}} \sum{}' \int_{C_{1\pm}} \int_{C_{2\pm}} \cdots \int_{C_{n\pm}} \cdot \prod_{i=1}^{n} u_1^{\sum_{m=1}^{N} \epsilon_{imkm}} h_{i\pm}(u_i)\theta(u_1, u_2, u_n)\, du_1\, du_2 \cdots du_n. \quad (18)$$

By analogy to (6) and (7), and differentiating (4) with respect to x, the right side of (18) is seen to be equal to

$$(-1)^k \left(\frac{1}{2}\right)^{\sum_{m=1}^{N} k_m \delta_{r_m s_m}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \cdot \prod_{i=1}^{n} f_i^{\left(\sum_{m=1}^{N} \epsilon_{imkm}\right)}(x_i) p(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n \quad (19)$$

thus yielding (3).

A Special Case and Its Applications

Consider the familiar situation where $n = 2$, and let ρ denote the crosscorrelation coefficient of x_1 and x_2. Then (3) yields

$$\frac{\partial^k R}{\partial \rho^k} = \overline{f_1^{(k)}(x_1) f_2^{(k)}(x_2)}. \quad (20)$$

Suppose that x_1 and x_2 are values of a stationary Gaussian time series $x(t)$ whose autocorrelation function is $\rho(\tau)$. x_1 is taken at time t and x_2 at time $(t + \tau)$. $R(\tau)$ will denote the crosscorrelation function between the outputs

[4] Cramér, *op. cit.*, sec. 24.1.

of two zero-memory nonlinear devices whose inputs are $x(t)$ and $x(t+\tau)$, respectively.

Taking the particular case where $\overline{x(t)} = 0$, $\overline{x^2(t)} = 1$, and using (1),[5]

$$\frac{\partial^k R(\tau)}{\partial \rho(\tau)^k} = \overline{f_1^{(k)}[x(t)] f_2^{(k)}[x(t+\tau)]} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{f_1^{(k)}(x_1) f_2^{(k)}(x_2) \exp\{-[x_1^2 + x_2^2 - 2\rho(\tau)x_1x_2]/2[1-\rho^2(\tau)]\}}{2\pi\sqrt{1-\rho^2(\tau)}}\, dx_1\, dx_2. \tag{21}$$

Eq. (21) is particularly simple when the $f_i(x)$ are piecewise-polynomial functions and k is sufficiently high. Then the $f_i^{(k)}(x)$ consist entirely of δ functions of various orders and the integral can be easily evaluated.

It is often of interest to obtain the derivatives of a crosscorrelation function with respect to τ. It is convenient to break down such τ derivatives into a series of products of derivatives of $R(\tau)$ with respect to $\rho(\tau)$, and $\rho(\tau)$ with respect to τ, using

$$\frac{dR(\tau)}{d\tau} = \frac{\partial R(\tau)}{\partial \rho(\tau)} \cdot \frac{d\rho(\tau)}{d\tau}. \tag{22}$$

This enables the nonlinear devices to be treated independently of the shape of the input correlation function $\rho(\tau)$, using (21). Similarly, the derivatives of $\rho(\tau)$ with respect to τ do not involve the $f_i(x)$.

As an example, Cohen[6] shows that in general, for autocorrelation functions $R(\tau)$, the limiting behavior of the corresponding power spectrum $\Phi(\omega)$ is given by:

$$\lim_{\omega\to\infty} \omega^2 \Phi(\omega) = -\frac{1}{\pi} \frac{dR(\tau)}{d\tau}\bigg|_{\tau=0+}$$
$$\lim_{\omega\to\infty} \omega^2\left[\omega^2\Phi(\omega) + \frac{1}{\pi}\frac{dR(\tau)}{d\tau}\bigg|\,\right]_{\tau=0+} = -\frac{1}{\pi}\frac{d^3R(\tau)}{d\tau^3}\bigg|_{\tau=0+} \tag{23}$$

and so on, where the derivatives are with respect to τ.

Another application of (20) is in deriving Bussgang's interesting result[7] that the crosscorrelation function between the input and the output of a nonlinear device driven by Gaussian noise has the same shape as the input autocorrelation function. In this case $f_1(x) = x$ and $f_2(x)$ is arbitrary. Then $f_1'(x)$ is unity, and all higher derivatives of $f_1(x)$ are zero. Putting this into (21) and evaluating the integral,

$$\frac{\partial^k R(\tau)}{\partial \rho(\tau)^k} = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \frac{f_2'(x) \exp(-x^2/2)}{\sqrt{2\pi}}\, dx & k = 1 \\ 0 & k > 1. \end{cases} \tag{24}$$

We find easily

$$R(\tau) = \rho(\tau) \int_{-\infty}^{+\infty} \frac{x f_2(x) e^{-x^2/2}}{\sqrt{2\pi}}\, dx \tag{25}$$

thus yielding Bussgang's result. Unlike Bussgang's theorem, (20) cannot be generalized to hold for probability distributions other than Gaussian.[8-10]

Some Simple Autocorrelation Examples [for $\overline{x(t)} = 0$, $\overline{x^2(t)} = 1$]

Hard Limiter

Van Vleck's well-known result on the autocorrelation function of the output of a hard limiter[11] can be derived very simply, using (21). If

$$f_1(x) = f_2(x) = \begin{cases} 1; & x \geq 0 \\ -1; & x < 0 \end{cases} \tag{26}$$

then $f_1^{(1)}(x)$ and $f_2^{(1)}(x)$ are first-order δ functions of area 2, at $x = 0$.

Substituting in (21) and integrating,

$$\frac{\partial R(\tau)}{\partial \rho(\tau)} = \frac{2}{\pi\sqrt{1-\rho^2(\tau)}}. \tag{27}$$

When $\rho(\tau) = 0$, $R(\tau) = 0$. Thus

$$R(\tau) = \frac{2}{\pi}\int_0^{\rho(\tau)} \frac{d\rho(\tau)}{\sqrt{1-\rho^2(\tau)}} = \frac{2}{\pi}\sin^{-1}[\rho(\tau)] \tag{28}$$

which is Van Vleck's result.

Linear Detector

Similarly, the autocorrelation function of the output of a linear detector can be easily found. If

$$f_1(x) = f_2(x) = \begin{cases} x; & x \geq 0 \\ 0; & x < 0 \end{cases} \tag{29}$$

then $f_1^{(2)}(x)$ and $f_2^{(2)}(x)$ are first-order δ functions of area unity at $x = 0$. Substituting in (21) and integrating:

$$\frac{\partial^2 R(\tau)}{\partial \rho(\tau)^2} = \frac{1}{2\pi\sqrt{1-\rho^2(\tau)}}. \tag{30}$$

Doubly-integrating (30) with the boundary conditions:

[5] I. N. Amiantov and V. I. Tikhonov, "The effect of normal fluctuations on typical nonlinear elements," *Bull. Acad. Sci. USSR*, pp. 33–42; April, 1956. Here, an autocorrelation case $[f_1(x) = f_2(x)]$ is studied by expanding the second-order joint Gaussian probability density of $x(t)$ and $x(t+\tau)$ in powers of $\rho(\tau)$, using Mehler's formula; see Bateman Manuscript Project, "Higher Transcendental Functions," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 2, p. 194, (22); 1953. They then integrate by parts to obtain the output correlation function in a similar series but do not recognize the simple form (21) for this series. Using this method it is not required that $f_1(x)$ be Laplace transformable, rather than our proof.

[6] R. Cohen, "Some Analytical and Practical Aspects of Wiener's Theory of Prediction," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Tech. Rep. No. 69, ch. 4, sec. 2; June 2, 1948.

[7] J. J. Bussgang, "Crosscorrelation Functions of Amplitude-Distorted Gaussian Signals," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Tech. Rep. 216, sec. 3; March 26, 1952.

[8] J. F. Barrett and D. G. Lampard, "An expansion for some second-order probability distributions and its application to noise problems," IRE Trans. on Information Theory, vol. IT-1, pp. 10–15; March, 1955.

[9] J. L. Brown, Jr., "On a cross-correlation property for stationary random processes," IRE Trans. on Information Theory, vol. IT-3, pp. 28–31; March, 1957.

[10] A. H. Nuttall, "Invariance of Correlation Functions under Nonlinear Transformations," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Quart. Progress Rep., p. 63; October 15, 1957.

[11] J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," McGraw-Hill Book Co., Inc., New York, N. Y., p. 58; 1950.

$$\left.\begin{aligned}\frac{\partial R(\tau)}{\partial \rho(\tau)} &= \left[\int_0^\infty f_1^{(1)}(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx\right]^2 = \frac{1}{4}\\ R(\tau) &= \left[\int_0^\infty \frac{xe^{-x^2/2}}{\sqrt{2\pi}}\,dx\right]^2 = \frac{1}{2\pi}\end{aligned}\right\}\text{ for } \rho(\tau) = 0 \quad (31)$$

we obtain:

$$\begin{aligned}R(\tau) &= \int_0^{\rho(\tau)}\left[\frac{1}{4} + \int_0^y \frac{dt}{2\pi\sqrt{1-t^2}}\right]dy + \frac{1}{2\pi}\\ &= \frac{1}{2\pi}\left\{\rho(\tau)\cos^{-1}[\rho(\tau)] + \sqrt{1-\rho^2(\tau)}\right\} \qquad (32)\end{aligned}$$

which is in agreement with Rice's result.[12]

Clipper

The relations derived independently by Robin[13] and Laning and Battin[14] for the autocorrelation function of the output of a clipper may also be found by this method. With a clipper characteristic:

$$f_1(x) = f_2(x) = \begin{cases} l; & l \le x\\ x; & -l \le x \le l\\ -l; & x \le l\end{cases} \qquad (33)$$

and $f_1^{(2)}(x)$ and $f_2^{(2)}(x)$ each are a pair of first-order δ functions at $x = -l$ and $x = l$, with areas 1 and -1, respectively. Substituting in (21) and integrating,

$$\frac{\partial^2 R(\tau)}{\partial \rho(\tau)^2} = \frac{\exp\left[-\dfrac{l^2}{1+\rho(\tau)}\right] - \exp\left[-\dfrac{l^2}{1-\rho(\tau)}\right]}{\pi\sqrt{1-\rho^2(\tau)}} \qquad (34)$$

which is Robin's result, for input noise of unit variance.

Smooth Limiter

Finally, Baum's recent interesting result[15] for the autocorrelation function of the output of a device having an error-function characteristic will be derived. With

$$f_1(x) = f_2(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2l^2}\,dt \qquad (35)$$

we have

$$f_1^{(1)}(x) = f_2^{(1)}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2l^2}. \qquad (36)$$

Substituting in (21):

$$\frac{\partial R(\tau)}{\partial \rho(\tau)} = \frac{1}{2\pi}\sqrt{\frac{\rho_1^2-\rho_2^2}{1-\rho^2(\tau)}}$$

$$\cdot\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\exp\left[-\dfrac{\rho_1x_1^2+\rho_1x_2^2-2\rho_2x_1x_2}{2(\rho_1^2-\rho_2^2)}\right]}{2\pi\sqrt{\rho_1^2-\rho_2^2}}\,dx_1\,dx_2\right\} \qquad (37)$$

where

$$\left.\begin{aligned}\rho_1 &= \frac{\{l^{-2}[1-\rho^2(\tau)]+1\}[1-\rho^2(\tau)]}{\{l^{-2}[1-\rho^2(\tau)]+1\}^2-\rho^2(\tau)}\\ \rho_2 &= \frac{\rho(\tau)[1-\rho^2(\tau)]}{\{l^{-2}[1-\rho^2(\tau)]+1\}^2-\rho^2(\tau)}\end{aligned}\right\} \qquad (38)$$

The term in braces in (37) must equal unity, since it is the integral of a second-order Gaussian probability density. Thus, from (38),

$$\begin{aligned}\frac{\partial R(\tau)}{\partial \rho(\tau)} &= \frac{1}{2\pi}\sqrt{\frac{\rho_1^2-\rho_2^2}{1-\rho^2(\tau)}}\\ &= \frac{1}{2\pi\sqrt{(1+l^{-2})^2 - l^{-4}\rho^2(\tau)}}. \qquad (39)\end{aligned}$$

Integrating and using the condition that when $\rho(\tau) = 0$, $R(\tau) = 0$,

$$\begin{aligned}R(\tau) &= \frac{l^2}{2\pi}\int_0^{\rho(\tau)}\frac{d\rho(\tau)}{\sqrt{(l^2+1)^2-\rho^2(\tau)}}\\ &= \frac{l^2}{2\pi}\sin^{-1}\left[\frac{\rho(\tau)}{1+l^2}\right] \qquad (40)\end{aligned}$$

which is in agreement with Baum's result.

[12] Rice, *op. cit.*, eq. (4.7–5).

[13] L. Robin, "The autocorrelation function and power spectrum of clipped thermal noise. Filtering of simple periodic signals in this noise," *Ann. Telecomm.*, vol. 7, pp. 375–387; September, 1952.

[14] J. H. Laning, Jr. and R. H. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y., p. 362, eq. (B-8); 1956.

[15] R. F. Baum, "The correlation function of smoothly limited Gaussian noise," IRE Trans. on Information Theory, vol. IT-3, pp. 193–197; September, 1957.

The Effect of Instantaneous Nonlinear Devices on Cross-Correlation*

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Summary—**If $X_1(t)$, $X_2(t)$ are two noises (stochastic processes), f and g are functions describing the action of two instantaneous nonlinear devices, we say that the (m, n) cross-correlation property holds in case the cross-correlation of $f(X_1(t_1))$ with $g(X_2(t_2))$ is proportional to the cross-correlation of $X_1(t_2)$ with $X_2(t_2)$, whenever f and g are polynomials of degrees not exceeding m and n, respectively. We take $m = \infty$ or $n = \infty$ to mean that f or g is any continuous function.**

The Barrett-Lampard expansion[2] of the second-order joint density of $X_1(t_1)$ and $X_2(t_2)$ is used to derive an expression for the cross-correlation of $f(X_1(t_1))$ and $g(X_2(t_2))$. This expression yields necessary and sufficient conditions for the validity of the cross-correlation property in three cases: $X_1(t)$ and $X_2(t)$ stationary, m, n unrestricted; $X_1(t)$ stationary, m, n unrestricted; $X_1(t)$ stationary, $n = 1$.

Examples are constructed with the help of special orthonormal polynomials illustrating the necessity and sufficiency of the conditions.

Introduction

Bussgang[1] showed that if one of a pair of stationary Gaussian processes is amplitude-distorted in an instantaneous nonlinear device, that the cross-correlations before and after distortion are proportional. In our terminology, this is the $(\infty, 1)$ cross-correlation property.

Barrett and Lampard[2] introduced a bi-orthonormal expansion method in a successful attempt to extend Bussgang's result to nonstationary processes with joint distribution whose bi-orthonormal coefficient matrix is diagonal. The latter is sufficient, but not necessary for the $(\infty, 1)$ cross-correlation property. Brown[3] found a necessary and sufficient condition for the $(\infty, 1)$ property when $X_1(t)$ and $X_2(t)$ are both stationary, and for a partially time-dependent $(\infty, 1)$ property when $X_1(t)$ is stationary, $X_2(t)$ nonstationary.

It is natural to ask whether a like property holds when both processes are distorted by instantaneous nonlinear devices and whether conditions can be relaxed if one or both devices are of restricted complexity (such as quadratic detectors).

We find that the results of Brown can be fully generalized in both of these directions (Theorems 1 and 2). We further discover (Theorem 3) that the partial time dependence allowed by Brown in the proportionality constant is absent when a stationary process is distorted and a nonstationary process is undistorted.

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[1] J. J. Bussgang, "Cross Correlation Functions of Amplitude Distorted Gaussan Signals," Res. Lab. of Electronics, M.I.T., Cambridge, Mass., Tech. Rep. No. 216; March 26, 1952.

[2] J. F. Barrett and D. G. Lampard, "An expansion for some second order probability distributions and its application to noise problems," IRE Trans. on Information Theory, vol. IT-1, pp. 10-15; March, 1955.

[3] J. L. Brown, Jr., "On a cross-correlation property for stationary random processes," IRE Trans. on Information Theory, vol. IT-3, pp. 28-31; March, 1957.

Analysis[4]

Let $p_{t_1,t_2}(x_1, x_2)$ be the joint density of two stochastic processes $X_1(t)$ and $X_2(t_2)$. If both are stationary, the time dependence involves $t_2 - t_1$ only. With Barrett and Lampard[2] we introduce the marginal densities $p_1(x_1)$, $p_2(x_2)$, sequences $\{\theta^{(1)}_{k,t_1}(x_1)\}$, $\{\theta^{(2)}_{k,t_2}(x_2)\}$ of polynomials orthonormal with respect to $p_1(x_1)$, $p_2(x_2)$, and the matrices $A_{t_1,t_2} = [a_{m,n;t_1,t_2}]$ such that

$$\frac{p_{t_1,t_2}(x_1, x_2)}{p_{t_1}(x_1)p_{t_2}(x_2)}$$

has the formal bi-orthonormal expansion

$$\sum_{m,n} a_{m,n,t_1,t_2}\, \theta^{(1)}_{m,t_1}(x_1)\, \theta^{(2)}_{n,t_2}(x_2)$$

We say the (m, n) cross-correlation property holds if for each polynomial f of degree $\leqq m$, each polynomial g of degree $\leqq n$, there is a constant $k(f, g)$ such that $\rho_{t_1,t_2}(f, g) = k(f, g)\, \rho_{t_1,t_2}$, where

$$\rho_{t_1,t_2}(f, g) = \frac{\operatorname{cov}\,(f(X_1(t_1)),\, g(X_2(t_2)))}{\sqrt{\operatorname{var}\,(f(X_1(t_1))\cdot \operatorname{Var}\,(g(X_2(t_2)))}}$$

$$\rho_{t_1,t_2} = \frac{\operatorname{cov}\,(X_1(t_1),\, X_2(t_2))}{\sqrt{(\operatorname{var} X_1(t_1))(\operatorname{var} X_2(t_2))}}. \tag{1}$$

The conditions derived below depend on a lemma stated here and proved in the Appendix.

Lemma: If

$$c_{u,t_1} = \int \theta^{(1)}_{u,t_1}(x_1)f(x_1)p_{1,t_1}(x_1)\, dx_1$$

$$d_{v,t_2} = \int \theta^{(2)}_{v,t_2}(x_2)g(x_2)p_{2,t_2}(x_2)\, dx_2$$

$$\text{for} \quad u, v = 0, 1, 2, \cdots,$$

where f and g are polynomials of degrees $\leqq m$ and $\leqq n$ respectively, then

$$\rho_{t_1,t_2}(f, g) = \frac{\sum_{u=1}^{M}\sum_{v=1}^{n} c_{u,t_1}d_{v,t_2}a_{u,v,t_1,t_2}}{\sqrt{\sum_{u=1}^{m} c^2_{u,t_1} \sum_{v=1}^{n} d^2_{v,t_2}}}. \tag{2}$$

[4] Questions of convergence are avoided in the formal calculations here. A rigorous treatment of the Barrett-Lampard expansion may be found in a forthcoming paper by the author in the *J. SIAM*.

This simple bilinear expression immediately yields

Theorem 1

If $X_1(t)$, $X_2(t)$ are stationary, then the (m, n) cross-correlation property holds if and only, if there is a time independent $m \times n$ matrix $Q = [q_{u,v}]_{m,n}$ with $q_{1,1} = 1$ such that the upper left $m \times n$ submatrix $B_{t_1,t_2} = [A_{t_1,t_2}]_{m,n}$ of A_{t_1,t_2} satisfies the equation $B_{t_1,t_2} = a_{1,1,t_1,t_2}Q$.

Proof: Since $X_1(t)$ and $X_2(t)$ are stationary, the Fourier coefficients of f and g are time-independent: $c_{u,t_1} = c_u$, $d_{v,t_2} = d_v$ for $u = 1, \cdots m$, $v = 1, \cdots n$. Hence

$$\rho_{t_1,t_2} f, g) = \frac{\sum_{u=1}^{m}\sum_{v=1}^{n} c_u\, d_v a_{u,v,t_1,t_2}}{\sqrt{\sum_{u=1}^{m} c_u^2 \sum_{v=1}^{n} d_v^2}}.$$

Note that $\rho_{t_1,t_2} = a_{1,1,t_1,t_2}$. If the matrix equation holds, then

$$\rho_{t_1,t_2}(f, g) = \frac{\sum_{u=1}^{m}\sum_{v=1}^{n} c_u\, d_v q_{uv}}{\sqrt{\sum_{u=1}^{m} c_u^2 \sum_{v=1}^{n} d_v^2}}\, \rho_{t_1,t_2}$$

$$= K(f, g)\rho_{t_1,t_2}.$$

(Actually, the correlations depend on $t_2 - t_1$ only).

Conversely, if $\rho_{t_1,t_2}(f, g) = K(f, g)\, \rho_{t_1,t_2}$, for polynomials f, g of degrees $\leqq m, n$, pick $f_i = \theta_i^{(1)}$, $g_k = \theta_k^{(2)}$ so that $c_u = \delta_{u,v}$, $d_v = \delta_{v,k}$, $\rho_{t_1,t_2}(f_i, g_k) = a_{j,k,t_1,t_2} = K(f_j, g_k)\, a_{1,1,t_1,t_2}$. Thus we have $q_{u,v} = K(f_u, g_v)$ for $u = 1, \cdots m$; $v = 1, \cdots n$ and $B_{t_1,t_2} = a_{1,1,t_1,t_2}Q$.

If only one of the processes is stationary, results are still obtainable.

Theorem 2

If $X_1(t)$ is stationary, then $\rho_{t_1,t_2}(f, g) = K_{t_2}(f, g)\, \rho_{t_1,t_2}$ for all polynomials f, g of degrees $\leqq m, n$ if and only if there exists an $m \times n$ matrix Q_{t_2} such that $B_{t_1,t_2} = a_{1,1,t_1,t_2}Q_{t_2}$.

Proof: We have $c_{u,t_1} = c_u$ for $u = 1, \cdots m$, so that

$$\rho_{t_1,t_2}(f, g) = \frac{\sum_{u=1}^{m}\sum_{v=1}^{n} c_u\, d_{v,t_2}a_{u,v,t_1,t_2}}{\sqrt{\sum_{u=1}^{m} c_u^2 \sum_{v=1}^{n} d_{v,t_2}^2}}.$$

If the matrix equation holds, then

$$\rho_{t_1,t_2}(f, g) = \frac{\sum_{u=1}^{m}\sum_{v=1}^{n} c_u\, d_{v,t_2}q_{t_2,u,v}}{\sqrt{\sum_{u=1}^{m} c_u^2 \sum_{v=1}^{n} d_{v,t_2}^2}}\, \rho_{t_1,t_2}$$

$$= K_{t_2}(f, g)\rho_{t_1,t_2}.$$

The converse is proved as above.

The time dependence of the proportionality constant in Theorem 2 above drops out, yielding the cross-correlation property of primary interest, in case it is only the stationary process X_1 which is nonlinearly distorted.

Theorem 3

If $X_1(t)$ is stationary, then the $(m, 1)$ cross-correlation property holds if and only if there is a m-vector $q = [q_v]m$ with $q_1 = 1$ such that the $m \times 1$ upper left vector b_{t_1,t_2} of A_{t_1,t_2} satisfies $b_{t_1,t_2} = a_{1,1,t_1,t_2}q$.

Proof: Since $X_1(t)$ is stationary, $c_{u,t_1} = c_u$ for $u = 1, \cdots m$. Since $X_2(t)$ is passed through at worst a linear device, $d_{v,t_2} = \delta_{v,1}$. Thus

$$\rho_{t_1,t_2}(f, g) = \frac{\sum_{u=1}^{m} c_u a_{u,1,t_1,t_2}}{\sqrt{\sum_{u=1}^{m} c_u^2}}.$$

If the vector equation holds, then

$$\rho_{t_1,t_2}(f, g) = \frac{\sum_{u=1}^{m} c_u q_u}{\sqrt{\sum_{u=1}^{m} c_u^2}}\, \rho_{t_1,t_2}.$$

The converse follows as in Theorem 1.

The results of Brown follow from Theorems 1 and 2 on setting $m = \infty$, $n = 1$.

Examples

1) The first example considered is where $X_1(t_1)$ and $X_2(t_2)$ have a joint Gaussian density

$$p_{t_1,t_2}(x_1, x_2) = \frac{1}{2\pi\sigma_{t_1}^{(1)}\sigma_{t_2}^{(2)}\sqrt{1 - \rho_{t_1,t_2}^2}} \cdot \exp\left\{-\frac{1}{2(1 - \rho_{t_1,t_2}^2)}\left[\frac{x_1^2}{\sigma_{t_1}^{(1)2}} - 2\rho_{t_1,t_2}\frac{x_1}{\sigma_{t_1}^{(1)}}\frac{x_2}{\sigma_{t_2}^{(2)}} + \frac{x_2^2}{\sigma_{t_2}^{(2)2}}\right]\right\}$$

with marginal densities

$$p_{t_1}(x_1) = \left(\frac{1}{2\pi\sigma_{t_1}^{(1)2}}\right)^{1/2} \exp\left\{-\frac{x_1^2}{2\sigma_{t_1}^{(1)2}}\right\}$$

$$p_{t_2}(x_2) = \left(\frac{1}{2\pi\sigma_{t_2}^{(2)2}}\right)^{1/2} \exp\left\{-\frac{x_2^2}{2\sigma_{t_2}^{(2)2}}\right\}.$$

The orthonormal polynomials are Hermite polynomials:

$$\theta_k^{(1)}(x_1) = \frac{1}{(2^k\cdot k!)^{1/2}} H_k\left(\frac{x_1}{\sigma_{t_1}^{(1)}\sqrt{2}}\right)$$

$$\theta_k^{(2)}(x_2) = \frac{1}{(2^k\cdot k!)^{1/2}} H_k\left(\frac{x_2}{\sigma_{t_2}^{(2)}\sqrt{2}}\right).$$

We then have

$$\frac{p_{t_1,t_2}(x_1, x_2)}{p_{t_1}(x_1)p_{t_2}(x_2)} = \frac{1}{\sqrt{1 - \rho_{t_1,t_2}^2}} \cdot \exp\left\{-\frac{1}{1 - \rho_{t_1,t_2}^2}\left[\rho_{t_1,t_2}^2\cdot\left(\frac{x_1^2}{2\sigma_{t_1}^{(1)2}} + \frac{x_2^2}{2\sigma_{t_2}^{(2)2}}\right) - 2\rho_{t_1,t_2}\frac{x_1x_2}{2\sigma_{t_1}^{(1)}\sigma_{t_2}^{(2)}}\right]\right\} = \sum_{n=0}^{\infty} \rho_{t_1,t_2}^n\theta_n^{(1)}(x_1)\theta_n^{(2)}(x_2).$$

Hence $a_{t_1,t_2,u,v} = \rho^u_{t_1,t_2}\delta_{u,v}$. It follows that if $X_1(t_1)$ and $X_2(t_2)$ are uncorrelated ($\rho_{t_1,t_2} = 0$), then so are $f(X_1(t_1))$ and $g(X_2(t_2))$ for all choices of f and g.

Choose $f(x) = g(x) = x^2$, so that

$$c_{0,t_1} = d_{0,t_2} = 1, \quad c_{1,t_1} = d_{0,t_2} = 0, \quad c_{2,t_1} = \sqrt{2}\sigma^{(1)^2}_{t_1},$$
$$d_{2,t_2} = \sqrt{2}\sigma^{(2)^2}_{t_2}, \quad c_{k,t_1} = d_{k,t_2} = 0 \quad \text{for} \quad k = 3, 4, \cdots .$$

We thus have

$$\rho_{t_1,t_2}(f, g) = 2\sigma^{(1)^2}_{t_1}\sigma^{(2)^2}_{t_2}\rho^2_{t_1,t_2}.$$

If $X_1(t_1)$ is stationary, then

$$\rho_{t_1,t_2}(f, g) = 2\sigma^{(1)^2}\sigma^{(2)^2}_{t_2}\rho^2_{t_1,t_2}.$$

If also $X_2(t_2)$ is stationary, then

$$\rho_{t_1,t_2}(f, g) = 2\sigma^{(1)^2}\sigma^{(2)^2}\rho^2_{t_1,t_2}.$$

The 2×2 matrix of a's is

$$\begin{bmatrix} \rho_{t_1,t_2} & 0 \\ 0 & \rho^2_{t_1,t_2} \end{bmatrix}.$$

Clearly the matrix equations and (2,2) cross-correlation conditions both reduce to $\rho_{t_1,t_2} = k$ and $\rho_{t_1,t_2} = k_{t_2}$ in the cases taken up in Theorems 1 and 2. It follows that if the cross-correlation is nonconstant in the doubly stationary case, the (m, n) property fails if $m \geqq 2$ and $n \geqq 2$.

2) The second example, or family of examples, is constructed with the help of Legendre polynomials. We find distributions satisfying: a) the (m, n) property but not the $(m, n + 1)$ or $(m + 1, n)$ properties, and b) the (∞, ∞) property.

It must be admitted that these examples are rather artificial.

Let $\{s_k(x)\}$ be the polynomials orthonormal on [0, 1] with respect to the rectangular distribution (unit weight function). Thus[5]

$$s_k(x) = \sqrt{2k+1}\,\frac{(2k)!}{2^k(k!)^2}\,P_k(2x - 1),$$

where the $P_k(x)$ are Legendre polynomials.

Let $T_k(y) = \int_0^y s_k(x)dx$. Since $s_0(x) = 1$, we have $T_0(y) = y$.

Clearly $T_k(0) = 0$ for all k. Moreover, $T_k(1) = \int_0^1 s_k(x)dx = \int_0^1 s_k(x)s_0(x)dx = \delta_{k0}$, so that $T_k(1) = 0$ for $k \geqq 1$. Since $s_k(x)$ is a polynomial, $T_k(y)$ is a polynomial, and we can write $T_k(y) = y(1 - y)\,U_k(y)$, where $U_k(y)$ is a polynomial of degree $k - 1$ for $k \geqq 1$.

Now set

$$F(x_1, x_2) = \sum_{i,j=0}^{\infty} b_{i,j}T_i(x_1)T_j(x_2),$$

where $b_{0,0} = 1, \quad b_{i,0} = b_{0,i} = 0, \quad i \geqq 1,$

and (x_1, x_2) is in the unit square.

[5] Follows from the normalization found in A. Erdelyi, "Higher Transcendental Functions," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 2, p. 179; 1953.

We wish to find conditions for $F(x_1, x_2)$ to be a joint cumulative distribution on the unit square. Note that $F(x_1, 0) = F(0, x_2) = 0$, and that

$$F(x_1, 1) = \sum_{i=0}^{\infty} b_{i0}T_i(x_1) = T_0(x_1) = x_1, \qquad F(1, x_2) = x_2.$$

Thus the marginal cumulative distributions $F_1(x_1)$ and $F_2(x_2)$ are rectangular, with densities $f_1(x_1) = f_2(x_2) = 1$. It remains only to show that $F(x_1, x_2)$ is jointly monotone in both variables on the unit square. If the joint density $f(x_1, x_2) = \partial^2F/\partial_{x_1}\partial_{x_2}$ exists in the interior of the unit square, the monotone character of $F(x_1, x_2)$ is insured by the nonnegativity of $f(x_1, x_2)$. Observe that

$$\frac{f(x_1, x_2)}{f_1(x_1)f_2(x_2)} = f(x_1, x_2) = 1 + \sum_{i,j=1}^{\infty} b_{ij}s_i(x_1)s_j(x_2).$$

Because of the alternating character of Legendre polynomials, not every choice of (b_{ij}) will make $f(x_1, x_2) \geqq 0$, so that a certain discretion must be exercised. However, a wide latitude is available. The classical inequality[6] $|P_k(x)| \leqq 1$ for $-1 \leqq x \leqq 1$ shows that $|s_k(x)| \leqq \sqrt{2k+1}\,(2k)!/2^k\cdot(k!)^2$ for $k \geqq 0$, $0 \leqq x \leqq 1$. Hence it is sufficient to so choose the b_{ij} that

$$1 + \sum_{i,j=1}^{\infty} \pm\, b_{ij}\sqrt{(2i+1)(2j+1)}\cdot\frac{(2i)!(2j)!}{2^{i+j}(i!)^2(j!)^2} \geqq 0$$

for all choices of sign. This crude condition suffices for the construction of the first example.

a) Let

$$b_{1,1} = \rho_{t_1,t_2} = \rho_{t_2-t_1}$$

and let

$$b_{m+1,n+1} = \alpha_{t_1,t_2} = \alpha_{t_2-t_1}$$

where $\rho_{t_2-t_1}$ and $\alpha_{t_2-t_1}$ are functions such that

$$\frac{\rho_{t_2-t_1}}{\alpha_{t_2-t_1}}$$

is nonconstant and

$$1 \pm 3\rho_{t_2-t_1} \pm \alpha_{t_2-t_1}\sqrt{(2m+3)(2n+3)} \cdot\frac{(2m+2)!(2n+2)!}{((m+1)!(n+1)!)^2 2^{m+n+2}}$$

is nonnegative for all four choices of sign. We set all the other $b_{ij} = 0$ for $i, j \geqq 1$. It follows from Theorem 1 that the (m, n) cross-correlation property holds, while the $(m + 1, n)$ and $(m, n + 1)$ properties fail.

The other type of example, in the opposite direction, is more elegant.

b) We begin with the little-known identity[7]

$$\sum_{k=1}^{\infty} \frac{2k+1}{k(k+1)}P_k(x)P_k(y) = -1 + 2\log 2 - \log\left[(1 - \min(x, y))(1 + \max(x, y))\right]$$

[6] *Ibid.*, p. 205.
[7] *Ibid*, p. 183.

valid for (x, y) in $[-1, 1] \times [-1, 1]$. Shifted to the unit square, this becomes, after some manipulation,

$$\sum_{k=1}^{\infty} \frac{2^{2k}(k!)^4}{k(k+1)[(2k)!]^2} s_k(x_1)s_k(x_2) = -1 - \log G(x_1, x_2),$$

where G is the Green's kernel

$$G(x_1, x_2) = \begin{cases} x_1(1 - x_2), & x_1 \geqq x_2 \\ x_2(1 - x_1), & x_1 \leqq x_2 \end{cases}$$

Since $0 \leqq G(x_1, x_2) \leqq 1$ in the unit square, $-\log G(x_1, x_2) \geqq 0$.

Now choose as coefficients

$$b_{i,j} = \rho_{t_2-t_1} \frac{2^{2i+1}(i!)^4}{i(i+1)((2i)!)^2} \delta_{i,j} \quad \text{for} \quad i, j \geqq 1.$$

Thus we have

$$\frac{f(x_1, x_2)}{f_1(x_1)f_2(x_2)} = f(x_1, x_2) = 1 + \sum_{i,j=1}^{\infty} b_{ij}s_i(x_1)s_j(x_2)$$

$$= 1 - 2\rho_{t_2-t_1} + 2\rho_{t_2-t_1}[-\log G(x_1, x_2)] \cdot$$

If the cross-correlation $\rho_{t_2-t_1}$ satisfies the inequalities $0 \leqq \rho_{t_2-t_1} \leqq \frac{1}{2}$ for all $t_2 - t_1$, then $f(x_1, x_2) \geqq 0$, and we have a legitimate distribution. (If $\rho_{t_2-t_1} = 0$, the example reduces to independent rectangular distributions). Since the conditions of Theorem 1 are satisfied for all m and all n, the above joint distribution has the (∞, ∞) cross-correlation property. In other words, each of two processes with the above joint distribution can be distorted through arbitrary instantaneous nonlinear devices without altering the time dependence of the cross correlation.

Appendix

We now derive the Lemma. Let

$$c_{u,t_1} = \int \theta_{u,t_1}^{(1)}(x_1)f(x_1)p_{1,t_1}(x_1)\, dx_1, \quad d_{v,t_2} = \int \theta_{v,t_2}^{(2)}(x_2)g(x_2)p_{2,t_2}(x_2)\, dx_2,$$

where f, g are polynomials of degrees m, n. Since $\theta_{0,t}^{(1)}(x) = \theta_{0,t}^{(2)}(x) = 1$, we have

$$c_{0,t_1} = \int f(x_1)p_{1,t_1}(x_1)\, dx_1 = E[f(X_1(t_1))], \quad d_{0,t_2} = \int g(x_2)p_{2,t_2}(x_2)\, dx_2 = E[g(X_2(t_2))].$$

Hence

$$f(x_1) - E[f(X_1(t_1))] = \sum_{u=1}^{m} c_{u,t_1}\theta_{u,t_1}^{(1)}(x_1), \quad g(x_2) - E[g(X_2(t_2))] = \sum_{v=1}^{n} d_{v,t_2}\theta_{v,t_2}^{(2)}(x_2),$$

and

$$E[(f(X_1(t_1)) - E[f(X_1(t_1))])(g(X_2(t_2)) - E[g(X_2(t_2))])]$$

$$= \iint p_{t_1,t_2}(x_1, x_2) \sum_{u=1}^{m}\sum_{v=1}^{n} c_{u,t_1}^{(1)}(x_1)\, d_{v,t_2}\theta_{v,t_2}^{(2)}(x_2)\, dx_1\, dx_2$$

$$= \sum_{u=1}^{m}\sum_{v=1}^{n} c_{u,t_1}\, d_{v,t_2} \iint \sum_{i,j} a_{i,j,t_1,t_2}p_{t_1}(x_1)p_{t_2}(x_2) \cdot \theta_{i,t_1}^{(1)}(x_1)\theta_{j,t_2}^{(2)}(x_2)\theta_{u,t_1}^{(1)}(x_1)\theta_{v,t_2}^{(2)}(x_2)\, dx_1\, dx_2$$

$$= \sum_{u=1}^{m}\sum_{v=1}^{n}\sum_{i,j} c_{u,t_1}\, d_{v,t_2}a_{i,j,t_1,t_2}\, \delta_{iu}\, \delta_{jv}$$

$$= \sum_{u=1}^{m}\sum_{v=1}^{n} c_{u,t_1}\, d_{v,t_2}a_{u,v,t_1,t_2}$$

on using the biorthonormal expansion for $p_{t_1,t_2}(x_1, x_2)$ and taking account of the orthonormality of $\{\theta_{m,t_1}^{(1)}(x_1)\}$ and $\{\theta_{m,t_2}^{(2)}(x_2)\}$, Similarly

$$E[(f(X_1(t_1)) - E[f(X_1(t_1))])^2]$$

$$= \int p_{t_1}(x_1)\left(\sum_{u=1}^{m} c_{u,t_1}\theta_{u,t_1}^{(1)}(x_1)\right)^2 dx_1$$

$$= \sum_{u=1}^{m}\sum_{v=1}^{n} c_{u,t_1}c_{v,t_1} \int p_{t_1}(x_1)\theta_{u,t_1}^{(1)}(x_1)\theta_{v,t_1}^{(1)}(x_1)\, dx_1$$

$$= \sum_{u=1}^{m} c_{u,t_1}^2$$

and

$$E[g(X_2(t_2)) - E[g(X_2(t_2))])^2] = \sum_{v=1}^{n} d_{v,t_2}^2$$

from which the desired result follows.

Some Properties of Nonbinary Error-Correcting Codes*

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Summary—An error-correcting code may be thought of as a ubset S_0 of points belonging to a set S in which a metric is defined uch that the distance between every pair of distinct points of S_0 s larger than some given number. In Hamming's original formuation, S was taken to be the set of all 2^n n-bit binary numbers nd the distance between a pair of binary numbers s and t was aken to be the number of bits of s which do not agree with the orresponding bits of t. In this note we shall take S to be the set of ll n-tuples in which each coordinate of an n-tuple can assume ne of k integral values: 0, 1, ..., $k-1$, with $k \geqq 2$. Properties f these nonbinary codes will be discussed.

AN error-correcting code may be thought of as a subset S_0 of points belonging to a set S in which a metric is defined such that the distance between very pair of distinct points of S_0 is larger than some iven number. In Hamming's original formulation,[1] S as taken to be the set of all 2^n n-bit binary numbers nd the distance between a pair of binary numbers s and was taken to be the number of bits of s which do not gree with the corresponding bits of t. In this note we hall take S to be the set of all n-tuples in which each oordinate of an n-tuple can assume one of k integral alues: 0, 1, $\cdots$, $k-1$, with $k \geqq 2$. These k values have he usual ordering and addition modulo k is assumed efined. For instance, if $k = 7$, then $4 + 6 = 3$ and $- 5 = 6 \pmod 7$. In our case a metric can rather aturally be assigned to S and a result of Hamming[1] can be xtended directly to group codes when each of the cordinates assumes one of a prime number of values .e., k is prime). This extension is implied, but not roved, in a recent paper by Ulrich.[2] A method is also ven for constructing certain single and double errorrrecting group codes for arbitrary k. For nongroup odes, we extend several inequalities on code size obtained r binary codes by Plotkin.[3] Finally, we obtain all closecked double error correcting codes for $k = 3$.

From here on, an n-tuple is called a *word* of length n d each coordinate of an n-tuple (word) is called a *letter*. he set of all words of length n in which each letter can sume one of k integral values 0, 1, $\cdots$, $k - 1$ is designted by $S_k(n)$. $S_k(n)$ has k^n members. Any subset of (n) is called a *code*.

Consider the k values of each letter as points placed ong a circle such that the circle is divided into k arcs equal length. Let the distance between two letters and y be the smallest number of arcs separating x and y. Let the distance between two words be the sum of the distances between their letters. Such a distance function is called a (*circular*) *metric* for $S_k(n)$. More precisely, if $s = (s_1, s_2, \cdots, s_n)$ and $t = (t_1, t_2, \cdots, t_n)$ are two words of $S_k(n)$, then ρ is the circular metric if

$$\rho(s, t) = \sum_{i=1}^{n} \rho(s_i, t_i)$$

where

$$\rho(s_i, t_i) = \text{Min } \{s_i - t_i, t_i - s_i\} \qquad (\text{mod } k).$$

The characteristics of a circular metric are exhibited by certain physical devices such as circular print wheels or ring counters.

As illustration let us observe several examples of ternary ($k = 3$) codes where each letter can take on one of three values 0, 1 and 2. The code C consisting of the following three 3-letter words in $S = S_3(3)$

0 0 0
1 1 1
2 2 2

is a ternary single error-correcting code. Should one of the words, say 111, be sent and a wrong word, say 121, be received, an encoding device could be instrumented to correct this single error since 121 is of a distance at least two away from the other words of the code.

We note that each member c of the code C has exactly six neighboring words in S with distance one away. These six words together with c form a closed ball of radius one about c. Since there can be not more than $3^3/(6 + 1)$, or less than 4 isolated balls of radius one in S, it follows that there can be not more than three words in any 3-letter ternary single error-correcting code. Thus we say the code C is *full* in S or the biggest possible in S.

We also note that S is an abelian group under addition modulo k and that the code C is a subgroup of S. C is therefore said to be a *group code* in S.

Following this line of thought we observe that the code D consisting of the following nine words in the set $T = S_3(4)$ of all 4-letter ternary words is again a single error-correcting code:

0 0 0 0
1 1 1 0
2 2 2 0
0 1 2 1
1 2 0 1
2 0 1 1
0 2 1 2
1 0 2 2
2 1 0 2

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[1] R. W. Hamming, "Error detecting and error-correcting codes," *ll. Sys. Tech. J.*, vol. 29, pp. 147–160; January, 1950.
[2] W. Ulrich, "Non-binary error correction codes," *Bell Sys. ch. J.*, vol. 36, pp. 1341–1388; November, 1957.
[3] M. Plotkin, "Binary Codes with Specified Minimum Distance," ore School of Elec. Eng., University of Pennsylvania, Philaphia, Pa.; 1952.

Since $3^4/(8 + 1)$ is exactly 9, it follows that no 4-letter single error-correcting code can have more than nine members. Also, D is a subgroup of T under addition modulo 3. Therefore D is a full group code.

There is an additional property of the code D which is not shared by the code C. Each member d of D belongs to a ball of radius 1 about d consisting of nine members of T. If we take the union of all these balls over all members of D, we find that this union is exactly T and not smaller. We say the code D is *close-packed* in T.

I. Theorems

We shall state and discuss several results which will be proved in Section III.

As given previously, let $S_k(n)$ be the set of all words of length n in which each letter can assume one of k integral values $0, 1, \cdots, k - 1$. A subset S_0 of $S_k(n)$ is said to be a single error-correcting code if the distance between every pair of distinct members of S_0 is 3 or greater, and in general, a subset S_0 of $S_k(n)$ is said to be an e-error-correcting code if the distance between every pair of distinct members of S_0 is $2e + 1$ or greater. For given k and n, we shall let $a_k(n, d)$ denote the size (the number of elements) of the largest e-error-correcting code in $S_k(n)$ and we shall let $b_k(n, d)$ denote the size of the largest e-error-correcting group code in $S_k(n)$ where $d = 2e + 1$. Then

Theorem 1

Let k be an odd prime. If $2n + 1$ is not a power of k, then

$$b_k(n + 1, 3) = kb_k(n, 3).$$

If $2n + 1$ is a power of k, then any group code with $b_k(n, 3)$ members is close-packed, and

$$b_k(n + 1, 3) = b_k(n, 3).$$

For $k = 2$, this is essentially Hamming's result.[4]

Single error-correcting group codes in which k is prime are completely characterized by Theorem 1. In the case k is not prime, Theorem 1 fails but we have

Theorem 2

Let m be some given positive integer and let C denote a single error-correcting group code.

1) For all odd k, $k > 2$, there is an $(m + r)$-letter code C such that C is close-packed and C has k^r members where $r = \frac{1}{2}(k^m - 2m - 1)$.

2) For k even, $k > 2$, there is an $(m + r)$-letter code C with $k^r/2$ members where $r = \frac{1}{2}(k^m - 2^m)$.

Theorems 1 and 2 indicate the behavior of single error-correcting group codes, but much less is known about double error-correcting codes. From (ii) of Theorem 2, we may show

[4] If $2n + 1$ is replaced by $n + 1$, then the case $k = 2$ is exactly Hamming's result.

Corollary 1: Let k be even, $k > 2$. For a given m, le $r = \frac{1}{2}(k^m - 2^m)$. Then an $(m + 2r)$-letter double error correcting group code with $k^r/2$ members can be con structed.

An example of such a double error-correcting code i as follows. Let $k = 6$, $m = 1$. Then, $r = \frac{1}{2}(k^m - 2^m) = 2$ By the corollary, a 5-letter double error-correcting grou code having 18 members can be constructed. Such a cod is

00000	20220	40440
02402	22022	42242
04204	24424	44044
11311	31531	51151
13113	33333	53553
15515	35135	55355.

So far we have been concerned with group codes. A more difficult problem is to construct full codes which may or may not be group codes. Plotkin gave[3] severa interesting bounds on the number of points in nongroup full binary codes. Extending his proofs, we may show the following two theorems.

Theorem 3[5]

For all pairs d, n such that $3d > 2n$,

$$a_3(n, d) \leqq 3d/(3d - 2n).$$

Some examples of this theorem are $a_3(4, 3) \leqq 9$, which we already know, $a_3(7, 5) \leqq 15$ and $a_3(10, 7) \leqq 21$.

Theorem 4

For all k,

$$a_k(2n, 2d) \geqq a_k(n, d)a_k(n, 2d).$$

As an example, we know $a_3(11, 10) = 3$, by Theorem 3 and from the next section, we shall see that $a_3(11, 5) = 3.$ Therefore, by this theorem, we have $a_3(22, 10) \geqq 3^7$.

The proofs of these two theorems will be left to the reader.

II. Some Remarks on Close-Packed Codes

In the binary case, Golay showed[6] that there is a close-packed, 23-letter, triple error-correcting code having 2^{12} members. It can be shown trivially that there is a close-packed, 5-letter, double error-correcting binary code having two members. Shapiro showed[7] (unpub lished) that there can be no close-packed double-erro correcting, binary code[8] for $n \neq 5$ and $n \neq 90$. Moreover

[5] The following extension to arbitrary k is probably true: $a_k(n, d$ $\leqq 4d/(4d - kn)$ when $4d > kn$ for k even, $a_k(n, d) \leqq 4kd/(4kd -$ $(k^2 - 1)n)$ when $4kd > (k^2 - 1)n$ for k odd.

[6] M. J. E. Golay, "Notes on digital coding," Proc. IRE, vol 37, p. 657; June, 1949.

[7] After this paper was submitted, the author learned that the unpublished material referred to will appear in a forthcoming pape by H. S. Shapiro and D. L. ~~Slotnick~~, "On the mathematical theor of error-correcting codes," *IBM J. Res.*

[8] S. P. Lloyd, "Binary block coding" *Bell Sys. Tech. J.*, vol 36, pp. 517–535; March, 1957. It is shown here that there is n close-packed code for $n = 90$.

he showed that there are at most a finite number of close-packed, e-error-correcting, binary codes over all n for $e \geqq 2$. We shall add a few facts here concerning k-valued, close-packed codes.

It follows from Theorems 1 and 2 that for each odd k, there is an infinite number of k-valued, close-packed, single error-correcting codes over all n, and it is clear that there can be no nontrivial close-packed, single error-correcting code for k even, $k > 2$. Let us consider next double error-correcting codes. The cases $k = 3$ and $k = 4$ are special and will be looked at first.

For $k = 3$, a closed ball of radius 2 about any point of $S_3(n)$ has exactly $2n^2 + 1$ members. Therefore, a necessary condition for a 3-valued, double error-correcting code to be close-packed is that $2n^2 + 1$ be a power of 3. That is, the diophantine equation

$$2n^2 + 1 = 3^x$$

be solvable in positive integral x and n. By chance this equation has been completely solved by Nagell,[9] presumably in no way connected with error correcting codes. The only possible solutions of this equation are

$$\begin{cases} n = 1 \\ x = 1 \end{cases} \quad \begin{cases} n = 2 \\ x = 2 \end{cases} \quad \begin{cases} n = 11 \\ x = 5. \end{cases}$$

The first two solutions are trivially sufficient, and each code has just one member. We therefore have

Remark 1: The only possible,[10] nontrivial, 3-valued close-packed, double error-correcting code is when $n = 11$.

For $k = 4$, we note that if we encode each value in binary so that the single letter 0 in the $k = 4$ system is encoded into the 2-letter binary word 00, 1 into 01, 2 into 11 and 3 into 10, then the circular metric in the $k = 4$ system is preserved in the corresponding binary system. Hence, every code in $S_4(n)$ is equivalent to a code in $S_2(2n)$. Therefore every close-packed code in $S_4(n)$ corresponds to a close-packed code in $S_2(2n)$. Since from previous discussion we know that there is no close-packed code in $S_2(2n)$, it follows that

Remark 2: There is no 4-valued, close-packed, double error-correcting code.

For $k > 4$, a closed ball of radius 2 about any point of $S_k(n)$ has $2n^2 + 2n + 1$ members. Therefore, for k prime, a necessary condition for a k-valued, $k > 4$, double error-correcting code to be close-packed is that $2n^2 + 2n + 1$ be a power of k. In other words, the diophantine equation

$$2n^2 + 2n + 1 = k^x$$

be solvable in positive, integral n, x and prime k.

This equation is trivially solvable for $x = 1$. For $x > 1$, it can be shown that there is an infinite number of solutions. However, this equation has at most a finite number of solutions for fixed k (see below), and at most a finite number of solutions[11] for fixed $x \geqq 3$.

Some known solutions to this equation are

$$\begin{cases} n = 3 \\ k = 5 \\ x = 2 \end{cases} \quad \begin{cases} n = 119 \\ k = 13 \\ x = 4 \end{cases} \quad \begin{cases} n = 4059 \\ k = 5741 \\ x = 2. \end{cases}$$

There is no code which satisfies the first solution. It is not known, although somewhat doubtful, whether there are codes which satisfy the other two solutions.

Extending a proof given by Shapiro,[7] it can be shown that

Remark 3: There can be at most a finite number of close-packed, e-error-correcting codes for each k, k prime, $k \geqq 2e + 1$ and $e \geqq 2$.

III. Definitions and Proofs

Let $S_k(n)$ be the set of all n-tuples $(x_1, x_2, \cdots, x_n)$ where x_i takes on integral values $0, 1, \cdots, k - 1$, for $i = 1, 2, \cdots, n$. Each member of $S_k(n)$ is called a *k-valued word of length n* or a *word* if no confusion arises. Each coordinate of a word is called a letter.

Now let a metric ρ be assigned to $S_k(n)$. We introduce the term *proximity* $P(S)$ of any subset S of $S_k(n)$ to be

$$P(S) = \text{Min}\, \{\rho(x, y) | x, y \in S, x \neq y\}.$$

Let d be a positive integer. Any subset of $S_k(n)$ whose proximity is not less than d is said to be an *e-error-correcting code* where $e = (d - 1)/2$ if d is odd, and is said to be an *e-error-detecting code* where $e = d/2$ if d is even. We shall call either an e-error-correcting or detecting code a d-code.

A d-code in $S_k(n)$ is said to be *full* if there is no d-code in $S_k(n)$ with more elements. For given n, d and k, let $a_k(n, d)$ denote the number of elements contained in a full d-code. A d-code is said to be *close-packed* if d is odd and if the union of closed balls of radius $(d - 1)/2$ about all members of the code exhausts $S_k(n)$.

Next let an operation + (mod k) be defined on $S_k(n)$ such that if s and t are two members of $S_k(n)$, each letter of $s + t$ is the sum modulo k of the corresponding letters of s and t. $S_k(n)$ is an abelian group under + (mod k). Any d-code on $S_k(n)$ is said to be a *group code* if it is a subgroup of $S_k(n)$. A group code is *full* if there is no group code having more elements. The number of elements in a full group code is denoted by $b_k(n, d)$.

Since the order of a finite group is a multiple of the order of every one of its subgroups and since the order of $S_k(n)$ is k^n, it follows that

Lemma 1: If k is a prime, $b_k(n, d)$ is a power of k.

[9] T. Nagell, "Sur l' impossibilité de quelques équations à deux indéterminées," *Norsk Matematisk Forenings Skrifter*, series 1, no. 13; 1923.

[10] A code satisfying the third solution has been constructed by Golay, *op. cit.*

[11] For a proof of this, see E. Landan and A. Ostrowski, "On the diophantine equation $ay^2 + by + c = dx^n$," *Proc. London Math. Soc.*, ser. 2, vol. 19, pp. 276–280, 1921.

It remains for us to specify the metric on $S_k(n)$. ρ is said to be the (*circular*) *metric* on $S_k(n)$ if for any pair of words $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ of $S_k(n)$,

$$\rho(x, y) = \sum_{i=1}^{n} \rho(x_i y_i)$$

where

$$\rho(x_i y_i) = \text{Min}\,\{x_i - y_i, y_i - x_i\} \pmod{k}.$$

The idea of the proofs for Theorems 1 and 2 runs as follows. We first show that a code can be constructed if a certain matrix can be found and then show how such a matrix can be obtained.

Definition 1: An m by r matrix $E(m, r)$ with entries a_{ij} is said to be *admissible* if when its columns are regarded as words of $S_k(m)$:

1) Every column is of distance more than one from the origin $(0, 0, \cdots, 0)$.
2) The sum or difference modulo k of any two columns shall be a word different from the origin.

For example, let $E_1(2, 3)$ and $E_2(2, 3)$ be matrices

$$E_1(2, 3) = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}, \qquad E_2(2, 3) = \begin{bmatrix} 0 & 1 & 4 \\ 3 & 2 & 3 \end{bmatrix}.$$

For $k = 5$, $E_1(2, 3)$ is not admissible since condition 1) is violated and $E_2(2, 3)$ is not admissible since condition 2) is violated.

Lemma 2: Let n be given and let m and r be any pair of positive integers such that $m + r = n$. Then for all odd k, an n-letter single error-correcting group code S having k^r members can be constructed if an admissible matrix $E(m, r)$ can be found.

Proof: Suppose $E(m, r)$ is found, the theorem will then be proved by construction. Denote by S the set of all n-letter words of the form.

$$(x_1 x_2 \cdots x_r y_1 y_2 \cdots y_m)$$

where $x_1 x_2 \cdots x_r$ are arbitrary but

$$y_i = \sum_{j=1}^{r} a_{ij} x_j \pmod{k}, \qquad i = 1, 2, \cdots, m,$$

with $a_{ij} \in E(m, r)$. S has k^r members.

Let $s_1 = (x_1 x_2 \cdots x_r y_1 y_2 \cdots y_m)$ and $s_2 = (u_1 u_2 \cdots u_r v_1 v_2 \cdots v_m)$ be two distinct words of S. $\rho(s_1, s_2) \geqq 3$ if three or more of the x_j's are different from the corresponding u_j's. Otherwise, there are two cases.

Case 1: $x_j = u_j$ except for $j = s, t$, $s \neq t$ for which $x_s \neq u_s$, $x_t \neq u_t$.

Let us suppose that the letter distances $\rho(x_s, u_s)$ and $\rho(x_t, u_t)$ are each exactly 1, for otherwise $\rho(s_1, s_2)$ would be larger than 2. Therefore, in this case,

$$x_s - u_s = 1 \quad \text{or} \quad k - 1 \pmod{k}$$

and

$$x_t - u_t = 1 \quad \text{or} \quad k - 1 \pmod{k}.$$

Since $E(m, r)$ is admissible, by condition 2 there exist p, q where $p \neq q$, $1 \leqq p, q \leqq m$ such that not both $a_{ps} + a_{pt}$ and $a_{qs} + a_{qt}$ are zero (mod k) and not both $a_{ps} - a_{pt}$ and $a_{qs} - a_{qt}$ are zero (mod k). We assert that either y_p is different from v_p or y_q is different from v_q. Suppose they are both identical, then

$$\begin{aligned} a_{ps}(x_s - u_s) + a_{pt}(x_t - u_t) &= 0 \\ a_{qs}(x_s - u_s) + a_{qt}(x_t - u_t) &= 0 \end{aligned} \pmod{k}$$

contradicting condition 2 of admissibility of $E(m, r)$. Therefore $\rho(s_1, s_2) \geqq 3$.

Case 2: $x_j = u_j$ except for $j = s$ for which $x_s \neq u_s$.

By condition 1 of admissibility of $E(m, r)$, either there are two members a_{ps}, $a_{qs} \neq 0$ or there is one member a_{ts}, $1 < a_{ts} < k - 1$.

In the first instance,

$$\begin{aligned} y_p - v_p &= a_{ps}(x_s - u_s) \\ y_q - v_q &= a_{qs}(x_s - u_s) \end{aligned} \pmod{k}.$$

First, suppose $x_s - u_s = 2 \neq k - 1 \pmod{k}$. Then, since k is odd, it is impossible for either $y_p - v_p = 0$ or $y_q - v_q = 0 \pmod{k}$. Next, suppose $x_s - u_s = 1$ or $k - 1 \pmod{k}$. Again it is impossible for either $y_p - v_p = 0$ or $y_q - v_q = 0 \pmod{k}$. Therefore, $\rho(s_1, s_2) \geqq 3$.

In the second instance,

$$y_t - v_t = a_{ts}(x_s - u_s) \pmod{k}.$$

Again since k is odd, if $x_s - u_s = 2 \neq k - 1 \pmod{k}$, $y_t - v_t \neq 0 \pmod{k}$. If $x_s - u_s = 1$ or $k - 1 \pmod{k}$, then $1 < y_t - v_t < k - 1$ implying $\rho(y_t, v_t) \geqq 2$ so that $\rho(s_1, s_2) \geqq 3$.

Thus, in either case, two distinct words of S are at a distance at least three apart. Since it can be checked that S is an abelian group, the proof follows.

In the case k is even, a slight modification in admissibility conditions is required.

Definition 2: An m by r matrix $E(m, r)$ is said to be *E-admissible* if $E(m, r)$ is admissible and

3) In every column j of $E(m, r)$, there is an element a_{ij} such that $2a_{ij} \neq 0 \pmod{k}$.

Lemma 3: Let m, r be defined such that $m + r = n$ where n is given. Let k be even. Then an n-letter single error-correcting group code S having k^r members exists if an E-admissible matrix $E(m, r)$ can be found.

Proof: The only part of the proof which is different from the proof for Lemma 2 is in Case 2 where $x_j = u_j$ except for $j = s$ for which $x_s \neq u_s$. Consider again two subcases.

Case 2.1: There are at least two members a_{ps}, $a_{qs} \neq 0$ in $E(m, r)$ such that either $2a_{ps} \neq 0 \pmod{k}$ or $2a_{qs} \neq 0 \pmod{k}$ by E-admissibility of $E(m, r)$. Suppose $2a_{ps} \neq 0 \pmod{k}$. Now

$$\begin{aligned} y_p - v_p &= a_{ps}(x_s - u_s) \\ y_q - v_q &= a_{qs}(x_s - u_s) \end{aligned} \pmod{k}.$$

First suppose $x_s - u_s = 2 \pmod{k}$. Then since $2\, a_{ps} \not\equiv 0 \pmod{k}$, $y_p - v_p \not\equiv 0 \pmod{k}$. Next suppose $x_s - u_s = 1$ or $k - 1 \pmod{k}$. Then $y_p - v_p \not\equiv 0$ and $y_q - v_q \not\equiv 0 \pmod{k}$. Therefore $\rho(s_1, s_2) \geqq 3$.

Case 2.2: There is exactly one nonzero member a_{ts} in the set $\{a_{is} \mid i = 1, 2, \cdots, m\}$. By E-admissibility of $E(m, r)$, it follows that $1 < a_{ts} < k - 1$ and $2a_{ts} \not\equiv 0 \pmod{k}$. Now

$$y_t - v_t = a_{ts}(x_s - u_s) \pmod{k}.$$

If $x_s - u_s = 2 \pmod{k}$, since $2a_{ts} \not\equiv 0 \pmod{k}$, it follows that $y_t - v_t \not\equiv 0 \pmod{k}$. If $x_s - u_s = 1$ or $k - 1 \pmod{k}$, then $1 < y_t - v_t < k - 1$. In either case $\rho(s_1, s_2) \geqq 3$. This concludes the proof.

Lemma 4: A code S, whenever constructable by application of Lemma 2, is close-packed (hence full) if $2n + 1$ is a power of k.

Proof: Each word s in S is contained in a ball consisting of $2n + 1$ words of radius 1 about s and different words of S are contained in disjoint balls. Thus, there are $k^n/(2n + 1) = k^r$ balls the union of which exhausts the set S.

Lemma 5: Let m, r be defined such that

$$k^m \geqq 2n + 1, \qquad k^{m-1} < 2n + 1$$

and

$$r = n - m.$$

An admissible matrix $E(m, r)$ having rm members exist for all *odd* k.

Proof: The lemma will be proved by construction. Let us consider, for each j, $j = 1, 2, \cdots, r$, the m-tuple $(a_{1j}, a_{2j}, \cdots, a_{mj})$ as a member of $S_k(m)$. Let S be a subset of $S_k(m)$ consisting of all members of $S_k(m)$ except the closed ball of radius 1 about $(0, 0, \cdots, 0)$. Define an equivalence relation $\sim$ on S such that if $s_1, s_2, \epsilon S$, $s_1 \sim s_2$ if $s_1 = s_2$ or $s_1 = s^{-1}_2$. Then any set of representatives (one from each equivalence class) has $(k^m - 2m - 1)/2$ words. Since

$$\frac{k^m - 2m - 1}{2} \geqq n - m = r,$$

choose r of these words arbitrarily to form a set of letters with rm members. Since this set of letters satisfies the two conditions of Def. 1, it is an admissible matrix and the proof follows.

Proof of Theorem 1: Define m such that

$$k^m \geqq 2n + 1 \quad \text{and} \quad k^{m-1} < 2n + 1.$$

Set $r = n - m$. Then by Lemmas 2 and 5, an n-letter group code S having k^r words exists.

Suppose first $2n + 1 = k^m$. Then by Lemma 4, S is close-packed so that $b_k(n, 3) = k^r$. Since

$$b_k(n + 1, 3) \leqq \frac{k^{n+1}}{2n + 3} < \frac{k^{n+1}}{2n + 1} = k^{r+1},$$

it follows by Lemma 1 that

$$b_k(n + 1, 3) = b_k(n, 3).$$

Next suppose $k^m > 2n + 1$ and $k^{m-1} < 2n + 1$. Should n be increased by 1, we still have

$$k^m \geqq 2n + 3 \quad \text{and} \quad k^{m-1} < 2n + 3,$$

so that m remains unchanged. Thus r is increased by 1, implying an $(n + 1)$-letter code having k^{r+1} words exists. Now

$$b_k(n + 1, 3) \leqq \frac{k^{n+1}}{2n + 3} = \frac{k^{r+2}k^{m-1}}{2n + 3} < k^{r+2}.$$

Therefore, it follows by Lemma 1 that $b_k(n + 1, 3) = k^{r+1}$, or

$$b_k(n + 1, 3) = kb_k(n, 3).$$

Lemma 6: Let k be even. For a given m, an E-admissible matrix having at least

$$m(\tfrac{1}{2}(k^m - 2^m - 2m))$$

members can be constructed.

Proof: Let us again consider, for each j, $j = 1, 2, \cdots, r$, the m-tuple $(a_{1j}, a_{2j}, \cdots, a_{mj})$ as a member of $S_k(m)$. Let S be the subset of $S_k(m)$ consisting of all members of $S_k(m)$ except 1) the closed ball of radius 1 about $(0, 0, \cdots, 0)$ and 2) all members s of $S_k(m)$ such that $s = s^{-1}$.

Again define an equivalence relation $\sim$ on S such that if $s_1, s_2, \epsilon S$, $s_1 \sim s_2$ if $s_1 = s_2$ or $s_1 = s^{-1}_2$. Then any set of representatives (one from each equivalence class) has $\frac{1}{2}(k^m - 2^m - 2m)$ members since there are exactly $2^m + 2m$ members of $S_k(m)$ which are not in S. The set of letters which make up members of S is then the E-admissible matrix in question.

Proof of Theorem 2, 1): Theorem 2, 1) follows from Lemmas 2 and 5.

Lemma 7: Let k be even. For some given n, let $b_k(n, 2)$ be the size of the largest code in which the distance between every pair of points is not less than 2 (*i.e.*, a single error-detecting code). Then $b_k(n, 2) = k^n/2$.

Definition 3. A matrix $E(m, r)$ is said to be *E′-admissible* if it satisfies condition 2 of Definition 1 and condition 3 of Definition 2.

Lemma 8: For a given m, there are at least $m(\frac{1}{2}(k^m - 2^m))$ members in an E'-admissible matrix.

Proof: The proof here is identical with the proof for Lemma 6 if we let the members of class 1 excluded from the set S in that proof be now included in S.

Proof of Theorem 2, 2) and Corollary 1: Let m be given and let $r = \frac{1}{2}(k^m - 2^m)$. Let S_r be an r-letter single

error-detecting code with $k^r/2$ members as given by Lemma 7. By Lemma 8, for the pair m, r in question, an E'-admissible matrix $E(m, r)$ can be constructed. For each $x = (x_1, x_2, \cdots, x_r)$ in S_r, define y_i such that

$$y_i = \sum_{j=1}^{r} a_{ij}x_j \pmod{k}, \qquad i = 1, 2, \cdots, m,$$

where each a_{ij} belongs to the E'-admissible matrix $E(m, r)$. Let S_{m+r} be the set of all $(m + r)$-letter words

$$(x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_m)$$

where $(x_1, x_2, \cdots, x_r)\ \varepsilon S_r$ and the y_i's are given above. Then S_{m+r} is a code satisfying the conditions of Theorem 2, 2).

Next let S_{m+2r} be the set of all $(m + 2r)$-tuples

$$(x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_m, x_1, x_2, \cdots, x_r)$$

where $(x_1, x_2, \cdots, x_r)\ \varepsilon S_r$ and the y_i's are determined b the linear relations above. S_{m+2r} has $k^r/2$ members an as in the proof of Theorem 1, we may show tha $\rho(s_1, s_2) \geqq 5$ for every pair s_1, $s_2\ \varepsilon S_{m+2r}$. Also, if s_1, s εS_{m+2r}, then $s_1 + s_2\ \varepsilon S_{m+2r}$. Therefore S_{m+2r} is a $(m + 2r)$-letter double error-correcting group code wit $k^r/2$ members, and the proof of Corollary 1 follows.

ACKNOWLEDGMENT

The writer is indebted to W. Ulrich and E. Wolma for helpful criticisms and discussions.

CORRECTION

Paul E. Green, Jr., author of "The Output Signal-to-Noise Ratio of Correlation Detectors," which appeared on pages 10–18 of the March, 1957 issue of these TRANSACTIONS, has requested the editors to make the following corrections to his paper.

In the first paragraph on page 12, the term u_i in the last line should be μ_i.

In (9) on the same page, $\mathrm{Re}([X(\omega)\ H(\omega)]^*[X(\omega)\ H(\omega)])$ should be $[X(\omega) \mid H(\omega) \mid^{2*} X(\omega)] + \mathrm{Re}([X(\omega)\ H(\omega)]^* [X(\omega)\ H(\omega)])$.

The right side of (11) on page 13 should be preceded by $| I(\Delta)|^2$.

In (12), on the same page, multiply the first denominator integrand term $X^2(\omega)\ \mathrm{Re}^2[H(\omega)]$ by a factor of two, and in (13) $N_2(\omega)$ should be replaced by $N_2(\omega) + \Delta)$.

The fifth from the last line in column two of page 13 should read:

For the bandpass filter, the reciprocal $\cdots$. Insert afte the last line: For the low-pass filter, the effective integra tion time is π/W_f.

The following should be added to footnote 8: By usin integration limits of $-\infty$ and $+\infty$ instead of 0 and ∞ one obtains a different definition of W_f, sometimes used which is the reciprocal of the effective integration tim for both types of filter.

In (25) on page 15, h_j^2 in the second term should b replaced by h_j, and ξ^2 should be ξ_i^2.

On page 16, the first line after (35), the equation refer ence should be (28), not (27).

On page 17, in the equation following (41), replac 1/8 with 1/2. In the second paragraph of column one the equation reference on the fourth line should be (25 instead of (3), and in (51), on the same page, the righ side should be multiplied by $| I(\Delta) |^2$.

The Effect of Noise Upon a Method of Frequency Measurement*

T. B. PICKARD†

Summary—**The effect of noise upon a method of information transmission and recovery is analyzed. The information is coded as the frequency shift of a carrier. The carrier is assumed to be transmitted over two channels which have 90° phase difference. The recovery of the information is accomplished in a new way by measuring the frequency shift by means of a "coherent cycle counter."**

A general expression is obtained for the expected frequency measurement of the coherent cycle counter in terms of the signal and noise autocorrelation functions. The percentage bias of the counter is shown to be a function only of the signal-to-noise ratios in the two channels. In the noise-free case, the expected frequency measurement is given by the centroid of the signal power spectrum. When no signal is present the average indicated frequency is zero, thereby effectively cancelling the average effect of the sources of noise in the system. For applications in which the transmitted frequency fluctuates about the carrier or reference frequency, the method is shown to be superior to both an axis crossing counter and an ideal phase differentiator when it is required to operate through zero frequency difference.

Introduction

A PROBLEM of frequency measurement arises whenever information is transmitted over a noisy channel as the frequency shift of a carrier.[1-3] Because of the random nature of the superimposed noise which arises in the system components as well as in the channel, the measured frequency is subject to random fluctuations. Additional fluctuations are present when the modulating signal itself is random. Consequently the frequency fluctuations must usually be smoothed to yield an average indicated frequency. When the modulating signal is sinusoidal, the desired output is, of course, the sine wave frequency. When the modulating signal is random, the desired output is some measure of the central tendency of the signal power spectrum.

One of the effects of noise being present with the signal at the input to a frequency measuring device is to introduce a bias into the average output. The bias for a particular device is defined as the difference between the mean output when noise is present and the mean output when no noise is present. In general, the bias will be a function of the signal-to-noise power, the average signal frequency, and the noise spectral shape. Thus, for example, the average output of an axis-crossing counter is the radius of gyration f_r of the signal power spectrum in the noise-free case. When noise is superimposed on the input signal, the average output is given by

$$f_{zc} = \left(\frac{P + K^2}{P + 1}\right)^{1/2} f_r \tag{1}$$

where K is the ratio of the radius of gyration of the noise power spectrum to the r. of g. of the signal spectrum, and P is the average signal-to-noise power. Consequently the percentage bias of the axis-crossing counter is

$$100\,\frac{\langle f_{zc}\rangle - f_r}{f_r} = \left[\left(\frac{P + K^2}{P + 1}\right)^{1/2} - 1\right] \times 100. \tag{2}$$

In a similar manner, the frequency bias of an ideal phase differentiator (a meter that measures frequency as the time rate of change of phase) when the input signal is a sine wave plus band-limited flat noise is[4]

$$100\,\frac{\langle f_{pd}\rangle - f_s}{f_s} = 100\,(K - 1)e^{-P} \tag{3}$$

where P and K have the same meaning as in (2) and f_s is the sine wave frequency.

From (2) and (3), it is seen that the bias of the two meters is zero when the parameter K is unity, that is, when the radius of gyration of the noise spectrum coincides with the signal radius of gyration. In other cases, the percentage bias is a function of both the signal-to-noise power and the value of the central noise frequency relative to the signal frequency.

Coherent Cycle Counting

In some applications it is required to measure the frequency shift of a transmitted signal whose instantaneous frequency f_T fluctuates about a carrier or reference frequency f_R. If the transmitted frequency is greater than the reference frequency, the frequency difference is regarded as positive; if less the difference is negative. The problem in this case is therefore not only to measure the difference between the transmitted and reference frequencies, but also to detect the algebraic sign.

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[1] P. M. Schultheiss, C. A. Wogrin, and F. Zweig, "Short-time frequency measurement of narrow-band random signals in the presence of wideband noise," *J. Appl. Phys.*, vol. 25, pp. 1025–1036; August, 1954.

[2] H. Steinberg, P. M. Schultheiss, C. A. Wogrin, and F. Zweig, "Short-time frequency measurement of narrow-band random signals by means of a zero-counting process," *J. Appl. Phys.*, vol. 26, pp. 195–201; February, 1955.

[3] G. R. Arthur, "Statistical properties of output of a frequency sensitive device," *J. Appl. Phys.*, vol. 25, pp. 1185–1195; September, 1954.

[4] S. O. Rice, "Statistical properties of a sine wave plus random noise," *Bell Sys. Tech. J.*, vol. 27, p. 129; January, 1948.

The following assumptions will be made with regard to the transmitted signal:

1) The transmitted frequency f_T lies in a frequency band about the reference frequency f_R.
2) It is possible to receive the transmitted signal in two channels which have 90° phase difference and in such a way that the noise sources in the two channels are uncorrelated.

Under these two assumptions, the transmitted information can be recovered by a new method of frequency measurement, described as "coherent cycle counting" in which the expected percentage bias depends only upon the signal-to-noise ratios in the two channels. In addition, the coherent cycle counter will detect the sign of the frequency deviation from the reference signal frequency.

A block diagram of the information transmission and recovery is given in Fig. 1. The coherent cycle counter has as its inputs the signal plus noise waveforms $X_1(t)$ and $X_2(t)$.

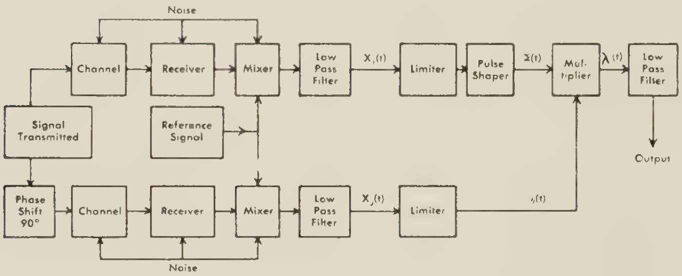


Fig. 1.

When the transmitted signal frequency exceeds the reference frequency, the phase of the signal in the second channel will be advanced 90° with respect to the signal phase in the first channel. When the transmitted frequency is less than the reference frequency, the phase difference of the signal in the two channels will be − 90°. To show this, suppose the transmitted signal is given by

$$S_T(t) = A_T \sin(\omega_R t \pm \omega_D t + \theta_1) \tag{4}$$

and the reference signal is given by

$$S_R(t) = A_R \sin(\omega_R t + \theta_2). \tag{5}$$

Then the mixed and filtered signal in the first channel will be proportional to $\cos(\pm \omega_D t + \theta_1 - \theta_2)$ or $\cos(\omega_D t \pm \theta_1 \mp \theta_2)$. On the other hand, the limiter input in the second channel will be proportional to $\cos(\pm \omega_D t + \theta_1 - \theta_2 - \pi/2)$ or $\cos(\omega_D t \pm \theta_1 \mp \theta_2 \mp \pi/2)$. Consequently the phase difference of the signal between the first and second channels is plus or minus 90° according as the transmitted frequency is greater or less than the reference frequency.

The limiter inputs in the two channels can therefore be represented by

$$X_1(t) = S_1(t) + N_1(t) \tag{6}$$

$$X_2(t) = S_2(t) + N_2(t) \tag{7}$$

where the frequency of the signal $S_1(t)$ is equal to the difference between the transmitted frequency and the reference frequency, $S_2(t)$ is equal to $S_1(t)$ with a 90° phase-shift, and $N_1(t)$ and $N_2(t)$ are the noise voltages present at the input to the limiters in the first and second channels, respectively.

The output of the pulse shaper is a train of positive and negative narrow pulses of equal area, the positions of which along the time axis correspond to zeros of the signal plus noise $X_1(t)$. Whether a pulse has positive or negative polarity depends upon the slope through zero of $X_1(t)$. The sign of a pulse is then reversed in the output of the multiplier if the limiter output in the second channel is negative. When the limiter output is positive the pulse passes through the multiplier with no change in polarity. The desired weighting function of the low-pass filter following the multiplier is given by

$$\begin{aligned} h(t) &= 1/2T, \qquad 0 \leq t \leq T \\ &= 0, \qquad \text{otherwise}. \end{aligned} \tag{8}$$

Consequently, the desired output at time t of the low-pass filter following the multiplier is given by the following functional:

$$F(X_1, X_2) = \frac{1}{2T} \int_{t-T}^{t} \delta(X_1) X_1'[2u(X_2) - 1]\, dt \tag{9}$$

where the prime notation denotes differentiation with respect to time, $\delta(X)$ is the Dirac delta function, and $u(X)$ is the unit-step function. The interpretation of (9) is as follows: let $N^+(t, T)$ denote the number of times the compound event

$$\{X_1(t) = 0 \quad \text{and} \quad X_1'(t) \cdot X_2(t) > 0\} \tag{10}$$

occurs in the interval $(t - T, t)$; similarly let $N^-(t, T)$ denote the number of occurrences of

$$\{X_1(t) = 0 \quad \text{and} \quad X_1'(t) \cdot X_2(t) < 0\} \tag{11}$$

in $(t - T, t)$. Then

$$N^+(t, T) = \int_{t-T}^{t} \delta(X_1) X_1' v(X_1', X_2)\, dt \tag{12}$$

and

$$N^-(t, T) = \int_{t-T}^{t} \delta(X_1) X_1' w(X_1', X_2)\, dt \tag{13}$$

for almost all realizations $X_1(t)$, $X_2(t)$, where

$$v(X_1', X_2) = u(X_1') + u(X_2) - 1 \tag{14}$$

$$w(X_1', X_2) = u(X_1') - u(X_2). \tag{15}$$

A rigorous proof of (12) and (13) can be made through the use of the Schwartz theory of derivatives.[5] For ou

[5] L. Schwartz, "Theorie des Distributions," Hermann, Paris, France, 1950.

poses it suffices to note that the integrands of (12) (13) are zero everywhere for almost all realizations $X_1(t)$ and $X_2(t)$, except for unit impulses coinciding time with positive and negative pulses, respectively, the multiplier output.

rom (12)–(15), the functional (9) can be expressed as

$$F(X_1, X_2) = \frac{1}{2T}[N^+(t, T) - N^-(t, T)]. \tag{16}$$

When the weighting function of the low-pass filter is en by (8), the frequency indication at time t is

$$f_{cc}(t, T) = \frac{1}{2T}[N^+(t, T) - N^-(t, T)]. \tag{17}$$

For stationary processes, the average indicated quency is independent of the time of observation t t depends on the time constant T of the counter. Taking ensemble average of (17) one finds for ergodic processes

$$\langle f_{cc}(T)\rangle = \frac{1}{2T}[\langle N^+(T)\rangle - \langle N^-(T)\rangle]. \tag{18}$$

The required ensemble averages $\langle N^+(T)\rangle$ and $\langle N^-(T)\rangle$ be found by calculating the expectations of the ctionals (12) and (13), respectively. For simplicity notation let $x_1 = X_1(t)$, $x_2 = X_1'(t)$, and $x_3 = X_2(t)$. en

$$\langle N^+(T)\rangle = \int_R \int_{t-T}^{t} \delta(x_1)x_2 v(x_1, x_3)\, dt\, d\Phi(x_1, x_2, x_3) \tag{19}$$

$$\langle N^-(T)\rangle = \int_R \int_{t-T}^{t} \delta(x_1)x_2 w(x_1, x_3)\, dt\, d\Phi(x_1, x_2, x_3) \tag{20}$$

ere $\Phi(x, y, z)$ = joint probability $\{x_1 < x, x_2 < x_3 < z\}$ and R is the three-dimensional probability ace of the random variables x_1, x_2, and x_3.

Before (19) and (20) can be evaluated, one needs to ow the statistics of the extracted signal plus noise veforms $X_1(t)$ and $X_2(t)$. The analysis is conveniently ided into two cases: first, when the extracted signal is ndom, and second, when the extracted signal is a sine ve.

se 1—Random Transmitted Signal

he first case assumes that the extracted signals $S_1(t)$ d $S_2(t)$ appearing at the limiter inputs in the first and ond channels, respectively, are stationary normal ndom variables with power spectral density $W_s(f)$. The ise voltages $N_1(t)$ and $N_2(t)$ at the limiter inputs are o assumed to be normally distributed with spectral nsities $W_1(f)$ and $W_2(f)$, respectively, and are unrelated to each other and to the extracted signals.

Assuming that the first order moments of $S_i(t)$ and (t) $i = 1, 2$, are all zero, the joint probability density the random variables x_1, x_2, x_3 is given by[6]

$$d\Phi(x_1, x_2, x_3) = (2\pi)^{3/2} \mid M \mid^{-1/2} \cdot \exp\left\{\frac{-1}{2\mid M\mid}\sum_{i,j=1}^{3} M_{ij}x_ix_j\right\} dx_1\, dx_2\, dx_3 \tag{21}$$

where the M_{ij} are the cofactors of the moment matrix $M = \| \mu_{ij} \|$ $\mid M \mid$ = determinant of M, and μ_{ij} = covariance of x_i and x_j.

The elements μ_{ij} of the moment matrix M can be determined by the use of Rice's expansion method[7] or by the use of the spectral representation formula.[8] When these calculations are performed, the moment matrix M turns out to be

$$M = \begin{Vmatrix} \psi_0 + \alpha_0 & 0 & 0 \\ 0 & -\psi_0'' - \alpha_0'' & \mp 2\pi\psi_0 f_c \\ 0 & \mp 2\pi\psi_0 f_c & \psi_0 + \beta_0 \end{Vmatrix} \tag{22}$$

where ψ_τ is the signal autocorrelation function, α_τ and β_τ are the noise autocorrelation functions in the first and second channels, respectively, and f_c is the centroid of the signal power spectrum; that is,

$$\psi_\tau = E\{S_1(t)S_1(t+\tau)\} = E\{S_2(t)S_2(t+\tau)\} = \int_0^\infty W_s(f)\cos 2\pi f\tau\, df \tag{23}$$

$$\alpha_\tau = E\{N_1(t)N_1(t+\tau)\} = \int_0^\infty W_1(f)\cos 2\pi f\tau\, df \tag{24}$$

$$\beta_\tau = E\{N_2(t)N_2(t+\tau)\} = \int_0^\infty W_2(f)\cos 2\pi f\tau\, df \tag{25}$$

$$f_c = \int_0^\infty f W_s(f)\, df/\psi_0 \tag{26}$$

where the right-hand integrals result from the familiar Wiener-Khintchine theorem.[9] The plus or minus sign prefixing f_c in (22) depends upon whether the transmitted frequency is greater or less than the reference frequency, respectively. From (22) it is apparent that $M_{12} = M_{13} = 0$.

When (21) is inserted in (19) and (20), the integration with respect to x_1 yields

$$\langle N^+(T)\rangle = (2\pi)^{-3/2} \mid M \mid^{-1/2} \cdot \int_{t-T}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 v(x_2, x_3)H(x_2, x_3)\, dx_2\, dx_3\, dt \tag{27}$$

and

$$\langle N^-(T)\rangle = (2\pi)^{-3/2} \mid M \mid^{-1/2} \cdot \int_{t-T}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 w(x_2, x_3)H(x_2, x_3)\, dx_2\, dx_3\, dt \tag{28}$$

[6] H. Cramer, "Mathematical Methods of Statistics," Princeton iversity Press, Princeton, N. J., p. 311; 1946.

[7] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, pp. 306–310; July, 1944.

[8] J. L. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y., p. 527; 1953.

[9] N. Wiener, "Generalized harmonic analysis," *Acta. Math.*, vol. 55, pp. 117–258; 1930.

where we have set

$$H(x_2, x_3) = \exp\left\{\frac{-1}{2\,|\,M\,|}(M_{22}x_2^2 + M_{33}x_3^2 + 2M_{23}x_2x_3)\right\}. \quad (29)$$

Recalling the definitions of $v(x_2, x_3)$ and $w(x_2, x_3)$ from (14) and (15), and noting that $H(-x_2, -x_3) = H(x_2, x_3)$ and $H(-x_2, x_3) = H(x_2, -x_3)$, (27) and (28) can be simplified as follows:

$$\langle N^+(T)\rangle = 2(2\pi)^{-3/2}\,|\,M\,|^{-1/2} \cdot \int_{t-T}^{t}\int_0^\infty\int_0^\infty x_2 H(x_2, x_3)\,dx_2\,dx_3\,dt \quad (30)$$

$$\langle N^-(T)\rangle = 2(2\pi)^{-3/2}\,|\,M\,|^{-1/2} \cdot \int_{t-T}^{t}\int_0^\infty\int_0^\infty x_2 H(-x_2, x_3)\,dx_2\,dx_3\,dt. \quad (31)$$

Eqs. (30) and (31) are similar to multiple integrals evaluated in the statistical study of certain nonlinear devices.[10,11] Thereupon the expected number of positive impulses counted in an interval of length T is

$$\langle N^+(T)\rangle = \frac{|\,M\,|\,T}{2\pi\sqrt{M_{22}}(\sqrt{M_{22}M_{33}} + M_{23})} \quad (32)$$

and the expected number of negative impulses is

$$\langle N^-(T)\rangle = \frac{|\,M\,|\,T}{2\pi\sqrt{M_{22}}(\sqrt{M_{22}M_{33}} - M_{23})}. \quad (33)$$

Consequently the average indicated frequency of the coherent cycle counter, after insertion of (32) and (33) into (18), is seen to be

$$\langle f_{cc}\rangle = \frac{-\,|\,M\,|\,M_{23}}{2\pi\sqrt{M_{22}}(M_{22}M_{33} - M_{23}^2)}. \quad (34)$$

When the appropriate cofactors in (19) are evaluated and inserted into (34), the average frequency indication of the coherent cycle counter is

$$\langle f_{cc}\rangle = \pm f_c\sqrt{\frac{P_1P_2}{(P_1+1)(P_2+1)}} \quad (35)$$

where $P_1 = \psi_0/\alpha_0$ and $P_2 = \psi_0/\beta_0$ denote the signal-to-noise ratios in the two channels, respectively.

In the absence of noise in both channels, (35) shows that the average indicated frequency of the coherent cycle counter is given by the centroid of the signal power spectrum.

When no signal is transmitted, the cofactor M_{23} in (22) vanishes, making $\langle N^+(T)\rangle = \langle N^-(T)\rangle$ in (32) and (33). Thus in the no-signal case, the coherent cycle counter has the desirable property that its average output is zero. This result is independent of the spectral shape of the noise waveforms in the two channels.

In the absence of noise in both channels, one can verify from (32) and (33) that the sum of the expected number of positive and negative pulses in the output of the multiplier reduces to Rice's result for the expected number of zeros of a random process:[12]

$$\langle N^+(T)\rangle + \langle N^-(T)\rangle = \frac{T}{\pi}\sqrt{\frac{-\psi_0''}{\psi_0}}, \quad \text{noise-free case.} \quad (36)$$

Eq. (35) shows that the percentage frequency bias depends only upon the signal-to-noise ratios in the two channels. Furthermore, since the phase difference in the two channels is $+90°$ or $-90°$ according as the transmitted frequency is greater or less than the reference frequency, the coherent cycle counter also acts as a sign detector, giving an average positive or negative voltage according to sign.

Case II—Sine Wave Signal

When the incoming signal is a sine wave

$$S_1(t) = A\sin\omega t \quad (37)$$

the joint density of the random variables x_1, x_2, and x_3 defined above is given by[6]

$$d\Phi(x_1, x_2, x_3) = (2\pi)^{-3/2}\,|\,M\,|^{-1/2} \cdot \exp\left\{\frac{-1}{2\,|\,M\,|}\sum_{i,j=1}^{3} M_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j)\right\} \cdot dx_1\,dx_2\,dx_3 \quad (38)$$

where $\bar{x}_1 = A\sin\omega t$, $\bar{x}_2 = A\omega\cos\omega t$, and $x_3 = \pm A\cos\omega t$.

The moment matrix is

$$M = \begin{Vmatrix} \alpha_0 & 0 & 0 \\ 0 & -\alpha_0'' & 0 \\ 0 & 0 & \beta_0 \end{Vmatrix} \quad (39)$$

where α_τ and β_τ are given by (24) and (25), respectively.

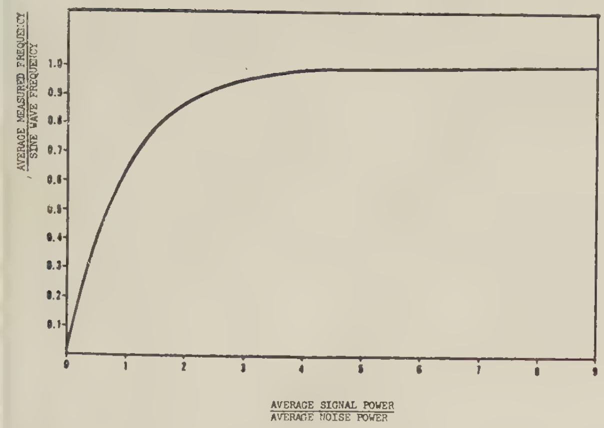
The average value of (9) gives the expected indicated frequency:

$$\langle f_{cc}\rangle = (2T)^{-1}\int_0^T \iiint_{-\infty}^{+\infty} \delta(x_1)x_2(2u(x_3) - 1)(2\pi)^{-3/2}\,|\,M\,|^{-1/2} \cdot \exp\left\{\frac{-1}{2\,|\,M\,|}[M_{11}(x_1 - \bar{x}_1)^2 + M_{22}(x_2 - \bar{x}_2)^2 + M_{33}(x_3 - \bar{x}_3)^2]\right\}dx_1\,dx_2\,dx_3\,dt. \quad (40)$$

[10] S. O. Rice, "Mathematical analysis of random noise (concluded)," *Bell Sys. Tech. J.*, vol. 24, pp. 67–69; January, 1945.

[11] J. H. Laning, Jr., and R. H. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y., Appendix A; 1956.

[12] Rice, "Mathematical analysis of random noise (concluded)," *op. cit.*, p. 54.



g. 2—Frequency measurement of coherent cycle counter for sine wave signal plus Gaussian noise.

The integration in the x's yields

$$\langle f_{cc}\rangle = (2T)^{-1}\sqrt{\frac{|M|}{2\pi M_{22}M_{33}}}$$

$$\cdot\int_0^T x_2 \exp\left(\frac{-M_{11}}{2|M|}\bar{x}_1^2\right)\phi\left(\frac{\sqrt{M_{33}}}{|M|}\bar{x}_3\right)dt \qquad (41)$$

here we have introduced the standard normal distribution

$$\phi(z) = (2\pi)^{-1/2}\int_{-z}^{z} e^{-1/2t^2}\,dt.$$

When the appropriate cofactors of (39) are inserted to (41), we have

$$\langle f_{cc}\rangle = \frac{\omega}{2T}\sqrt{\frac{P_1}{\pi}}\int_0^T \cos\omega t$$

$$\cdot\exp(-P_1\sin^2\omega t)\phi(\pm\sqrt{2P_2}\cos\omega t)\,dt \qquad (42)$$

here $P_1 = A^2/2\alpha_0$ is the average signal-to-noise power tio in the first channel and $P_2 = \mathring{A}^2/2\beta_0$ is the average gnal-to-noise in the second channel.

As the integrand of (42) is periodic with period $1/2f$, here $f = \omega/2\pi$, the integration can be simplified:

$$\frac{\langle f_{cc}\rangle}{f} = \sqrt{\frac{P_1}{\pi}}\int_0^{\pi} dv\cos v\exp(-P_1\sin^2 v)\phi(\pm\sqrt{2P_2}\cos v)$$

$$\frac{\langle f_{cc}\rangle}{f} = \pm 2\sqrt{\frac{P_1}{\pi}}\int_0^1 dx\, e^{-P_1x^2}\phi(\sqrt{2P_2(1-x^2)}). \qquad (43)$$

Eq. (43) shows as in the case of a random signal that e average indicated frequency of the coherent cycle-ounter is independent of the shape of the noise power ectra and is a function only of the signal frequency and e signal-to-noise ratios P_1 and P_2 in the two channels. the no-signal case ($P_1 = P_2 = 0$), the average indicated equency is again zero.

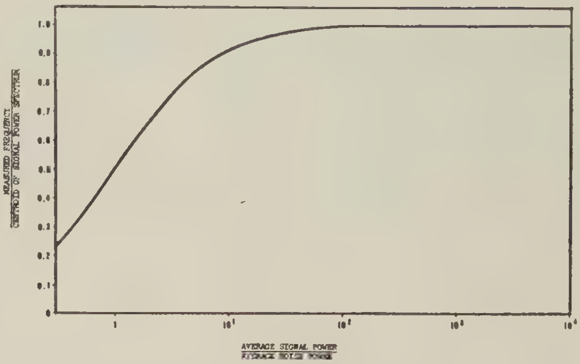


Fig. 3—Frequency measurement of coherent cycle counter for Gaussian signal plus Gaussian noise.

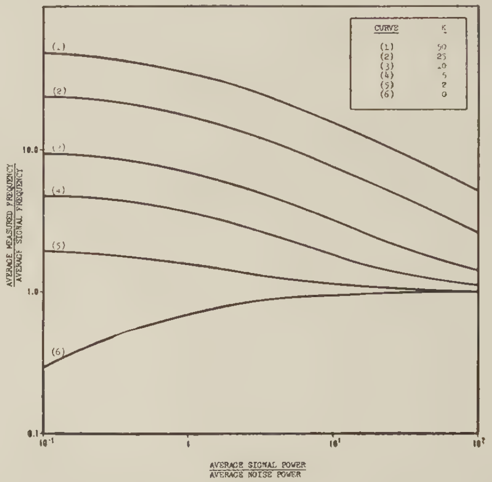


Fig. 4—Average frequency measurement of zero counter for Gaussian signal plus Gaussian noise. K = average noise frequency/average signal frequency.

Unfortunately when $P > 0$, (43) cannot be integrated in closed form, so that numerical methods have to be resorted to. Fig. 2 shows the results of the numerical integration for the case $P_1 = P_2$, where the ratio of average indicated frequency to signal frequency is plotted vs average signal-to-noise power.

Conclusions

When a random signal is transmitted, the coherent cycle counter can be compared with the axis crossing counter by means of (1) and (35). Eq. (35) shows that the bias of the coherent cycle counter can be made zero by inserting a gain equal to

$$\sqrt{\frac{(P_1+1)(P_2+1)}{P_1P_2}}.$$

The equivalent operation of the axis crossing counter requires knowledge of the noise spectrum as well as of the average signal-to-noise ratio. The expected frequency measurement of a random signal by the coherent cycle counter and the zero counter are given by Figs. 3 and 4, respectively.

Similarly, when a sine wave signal is transmitted, the gain required to eliminate the bias is dependent only on the average signal-to-noise ratios, as is shown by (43). For an idealized phase differentiator (2) shows that the gain adjustment is dependent both on the noise spectrum and the average signal-to-noise power. Figs. 2 and 5 give the expected measured frequency of a sine wave perturbed by Gaussian noise for the coherent cycle counter and the phase differentiator.

Finally, the coherent cycle counter also acts as a sign detector, that is, it specifies whether the average transmitted frequency is greater or less than the reference frequency. This fact allows the coherent cycle counter to operate through zero frequency difference between the transmitted and reference signals.

Acknowledgment

The author wishes to thank the Ryan Aeronautical Company for permission to publish the above material and M. A. Condie for many interesting discussions.

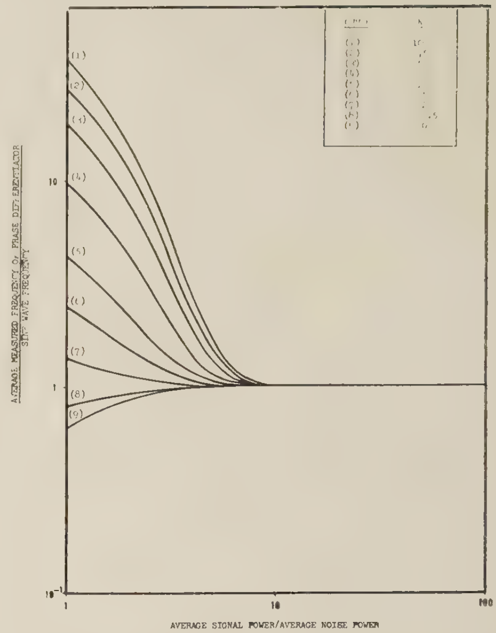


Fig. 5—Average frequency of sine wave plus band-limited "whit Gaussian noise. K = average noise frequency/sine wave frequen

Correspondence

A Comment on the Optimum Nonlinear Filtering of Balakrishnan and Drenick

In a recent paper, Balakrishan and Drenick derived optimum nonlinear filters applicable to strictly stationary discrete signals.[1] Their filtering procedure consists of three steps in cascade. First, a linear operation changes the input to a new process possessed of a white spectrum. A nonlinear zero memory operation is then performed. Finally, another linear filter restores the original spectrum to the output.

The above mentioned linear operations relating the input process x_n to the white process ζ_n can involve only present and past values of x_n and ζ_n if they are to be realizable. These operations are therefore given by Balakrishan and Drenick in the form

$$x_n = \sum_0^\infty a_j \zeta_{n-j} \qquad (1)$$

and

$$\zeta_n = \sum_0^\infty b_j x_{n-j}. \qquad (2)$$

That the representation (1) and its inverse (2) actually exist is an assumption basic to the referenced paper. No attempt was made there to explore the conditions under which (1) and (2) are valid, and if so, how the coefficients a_j and b_j are to be evaluated.

This note will indicate that (1) is valid under very general conditions, and that whenever (1) is true, (2) holds also. In addition, coefficients of (1) and (2) are determined.

The existence of (1) is assured if we eliminate from consideration inputs whose future can be predicted precisely from their past.[2] Such inputs are called deterministic, and are of little interest in filtering and prediction theory. Likewise, if an input is composed of a deterministic and nondeterministic part, we are not concerned with the deterministic (perfectly predictable) portion.

An input process is wholly nondeterm-istic in the sense just described [and (is valid] if and only if the spectral densi $\Phi(f)$ exists (without delta functions) and

$$\int_{-1/2}^{+1/2} \log \Phi(f)\, df > -\infty . \qquad ($$

This requirement is analogous to t Paley-Wiener condition needed for filteri and prediction of continuous paramet random processes.

A unique determination of the constar a_j is now possible in terms of log $\Phi($ Again, the proof is omitted here.[2] We ha

$$a_0 = \exp \frac{1}{2} \left\{ \int_{-1/2}^{+1/2} \log \Phi(f)\, df \right\} \qquad ($$

if the ζ_n have been normalized so that th mean square value is unity. The remaind of the a_j are defined through the relatic ship

$$\sum_0^\infty a_j z^j = \exp \left[c_0 + 2 \sum_1^\infty c_k z^k \right] \qquad ($$

where z is complex with $|z| < 1$, and t c_k are computed from

$$c_k = \frac{1}{2} \int_{-1/2}^{+1/2} e^{2\pi i k f} \log \Phi(f)\, df. \qquad ($$

[1] A. V. Balakrishan and R. Drenick, "On optimum nonlinear extraction and coding filters," IRE Trans. on Information Theory, vol. IT-2, pp. 166–172; September, 1956.

[2] See J. L. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y., pp. 577–579; 1953.

To determine a given a_j from (5), it is ly necessary to differentiate the right-nd side of the equation j times in z, set $= 0$, and divide by $j!$. Note that a_j is a nction only of those c_k whose index $k \leq j$. erefore, finding the first m a_j means at only the first m c_k need be obtained.

The question of the number of a_j which ould be computed is answered as follows. cause ζ_n is an orthonormal sequence and converges in probability mean,

$$E \mid x_n \mid^2 = \sum_0^{\infty} \mid a_j \mid^2 \tag{7}$$

that a finite memory filter utilizing only first m coefficients incurs a mean uare error of

$$E \mid x_n \mid^2 - \sum_0^{m-1} \mid a_j \mid^2.$$

In practical applications, the residual error must be balanced against the additional complexity resulting from the use of a greater number of terms.

Turning now to (2), we note that ζ_n, ζ_{n-1}, ζ_{n-2} $\cdots$ constitutes a basis in the manifold of x_n, x_{n-1}, x_{n-2} $\cdots$. It follows that ζ_n must be a linear combination of those x_k for which $n \geq k$; this is exactly the form (2).

The coefficients b_j of (2) are most easily obtained in terms of the a_j. In the first place, multiplying (1) by ζ_{n-r} and taking the ensemble average yields

$$E(x_n \zeta_{n-r}) = a_r. \tag{8}$$

Performing the same operation on (2) and substituting (8) then gives

$$\sum_{j=0}^{k} b_j a_{k-j} = \delta_{0k} \tag{9}$$

for $k = 0, 1, 2, \cdots$. This set of equations can be solved, for example, by starting with $k = 0$ and then proceeding to equations of higher k consecutively. Thus, in each equation b_k appears in terms of b_{k-1}, b_{k-2}, $\cdots$ which have already been found.

To determine how many b_j are needed to approximate ζ_n satisfactorily, we again compute the error when m such b's are used in a finite memory filter. This error is

$$1 - \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} b_j b_k R(j - k) \tag{10}$$

where $R(p)$ is the correlation function $E(x_r x_{r+p})$.

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PGIT News

FILIATES OF THE IRE PRO-SSIONAL GROUP ON INFORMA-ON THEORY

The Affiliate Plan permits qualified non-mbers of IRE to become affiliated with tain Professional Groups without first ving to join the IRE itself. The details the plan are outlined in an article by . W. R. G. Baker, chairman of the ofessional Groups Committee, published the March, 1957 issue of these TRANSACTIONS.

To date, affiliation of the following ieties to the Professional Group on formation Theory has been approved.

- American Association for the Advancement of Science
- American Astronomical Society
- American Chemical Society
- American Documentation Institute
- American Geophysical Union
- American Institute of Electrical Engineers
- American Institute of Physics and its member Societies:
  - Acoustical Society of America
  - American Physical Society
  - American Association of Physics Teachers
  - Optical Society of America
  - Society of Rheology
- American Management Association
- American Mathematical Society
- American Meteorological Society
- American Physical Society
- American Psychological Association
- American Rocket Society
- American Society of Mechanical Engineers
- American Speech and Hearing Association
- American Statistical Association
- Armed Forces Communications and Electronics Association
- Association for Computing Machinery
- Audio Engineering Society
- Institute of the Aeronautical Sciences
- Institute of Mathematical Statistics
- Institute of Navigation
- Instrument Society of America
- Linguistic Circle of New York
- Linguistic Society of America
- Modern Language Association of America
- New York Academy of Sciences
- Operations Research Society of America
- Society for Industrial and Applied Mathematics
- Society of Motion Picture and Television Engineers

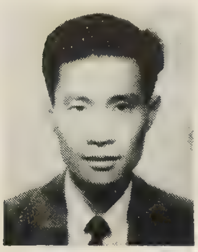
Additional information on the plan or on how to become an Affiliate can be obtained directly from IRE Headquarters.

DR. WILLIAM G. TULLER MEMORIAL AWARD

Awards of $250.00 will be made by the IRE Professional Group on Component Parts for the best papers submitted by a senior or graduate student on the subject of component parts. The subject may relate to operational theory, materials, construction, design, testing, or application of any electronic component. Papers must be submitted by December 31, 1958. Consult the Office of the Dean or IRE Faculty Advisor for additional information.

Contributors

C. Y. Lee (S '47—A '50—M '55) was born in Shanghai, China, on December 10, 1926. He received the B.E.E. degree from Cornell University in 1947, and the M.S.E.E. and Ph.D. degrees in mathematics from the University of Washington in 1949 and 1954, respectively.



C. Y. Lee

Dr. Lee is a research mathematician with Bell Telephone Laboratories, Whippany, N. J. His principal interest is in electronic switching systems and computer theory. He is on leave at the Institute for Advanced Study, Princeton, N. J., for the academic year 1957–1958.

He is a member of Sigma Xi and Eta Kappa Nu.

❖

Roy Leipnik was born in Los Angeles, Calif., May 6, 1924. After attending the University of California at Berkeley from 1941–1943, he received S.B. and S. M. degrees in mathematics from the University of Chicago in 1946 and 1948, respectively. He was a fellow at the Institute for Advanced Study, Princeton, N. J., from 1948 to 1950, and received the Ph.D. degree in mathematics from the University of California at Berkeley in 1950. From 1950 to 1957 he was on the mathematics faculty at the University of Washington, Seattle, and in 1955 was Fulbright Research Professor in mathematical physics at the University of Adelaide, Australia, where he worked on problems in stochastic processes in physics, operator decomposition, and Mach's principle.

R. Leipnik

Dr. Leipnik has been a consultant to the U. S. Naval Ordnance Test Station since 1956, where he has been engaged in research on stochastic processes and information theory. He is a member of Phi Beta Kappa, Sigma Xi, the American Mathematical Society, the Institute of Mathematical Statistics and serves on the board of directors of Decisional Control Associates, Inc.

❖

T. B. Pickard was born in San Diego, Calif., on August 16, 1925. He received the A.B. degree from San Diego State College in 1949, and the M.A. degree in mathematics from U.C.L.A. in 1958.

T. B. Pickard

From 1943 to 1946 he served in the Navy as a fire controlman, and from 1949 to 1954 he was engaged in actuarial work with a life insurance company in Los Angeles. The following three years h was employed by the Ryan Aeronautics Company in San Diego. He is presentl pursuing further studies at the Universit of California, Berkeley.

❖

Robert Price, for a photograph and bi ography please see page 60 of the Marcl 1958 issue of these Transactions.

❖

David Slepian (A '52) was born on Jun 30, 1923 in Pittsburgh, Pa. He attended th University of Michigan from 1941 to 194 and then entere the U. S. Arm Signal Corps. I 1946 he enrolled a Harvard Universit Cambridge, Mass where he receive the M.A. degree i 1947, and the Ph.D degree in physics i 1949. He studied i Europe for one yea during 1949–1950 a a Parker Fello from Harvard University.

D. Slepian

He joined the staff of Bell Telephon Laboratories in 1950 and has been engage in mathematical research in the fields c communications theory, switching theor and noise theory.

Dr. Slepian is a member of the America Mathematical Society, American Associa tion for the Advancement of Science, Sigm Xi, and the Society for Industrial an Applied Mathematics.

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Kunze, A. A.
Libby, R. L.
Moldoff, S.
Morelli, M.
Oot, R. F.
Schultz, J. L.
Sears, C. B.
Shiner, G. A.
Story, R. T.
Walker, R. K.
Winn, O. H.

Schenectady

Beers, R. F.
Chernoch, J.
Dodson, G. C., Jr.
Fanuele, F. J.
Fick, C. G.
Fulton, J. T.
Harris, H. D.
Jakowatz, C. V.
Mihran, T. G.
Pester, R. F.
Shuey, R. L.
Stutt, C. A.
Taylor, J. E.
Thomas, A. W.

Syracuse

Allen, A. E.
Applebaum, S. P.
Bartley, W. P.
Barton, L. M.
Berlin, R. D.
Brain, A. E.
Chapman, J. K.
Chynoweth, W. R.
Cross, K. B.
Dickey, F. R., Jr.
Edwards, K. A.
Egtvedt, M. D.
Eichel, J. H.
Ellis, S. S.
Fitzmorris, S. R.
Forney, J. F.
Fye, R. P.
Garber, S. M., Jr.
Goldman, S.
Grisetti, R. S.

Habermann, R., Jr.
Hoefer, W. G.
Hu, M. K.
Jacobs, G. B.
Lambert, J. M.
Lothes, R. N.
Markes, R. T.
Neelands, L. J.
Noble, M. L.
Reza, F.
Russell, J. B., Jr.
Shuart, O. H.
Varga, J. A.
Woods, C. R.
Woodward, G. F.

Western Massachusetts

Dulchinos, J.
Mahar, T. F.
Samsel, R. W.

Region 2

Long Island

Bailey, W. F.
Barret, J. P.
Bicknese, F. H.
Bifano, V. J.
Bloom, G. M.
Blum, R. J.
Bodow, J.
Boyd, P. C., Jr.
Brandinger, J. J.
Brockner, C. E.
Brownman, H. L.
Bruce, G. B.
Burgess, E. G., Jr.
Burrows, C. R.
Byron, M.
Campopiano, C. N.
Capon, J.
Chadwick, J. H., Jr.
Chako, N.
Chartoff, P.
Codina, J. G.
Cornetz, W.
Cowgill, D. E.
Crist, P. W.
Crosby, M. G.
Crowder, H. A.
Crowhurst, N. H.
Damast, M. A.
D'Angelo, H.
Darden, R. R., Jr.
Davidson, S.
Dean, M. A.
De Gennaro, R.
Derr, E. H., Jr.
Deutsch, S.
Di Toro, M. J.
Dolin, R.
Dong, H. L. H.
Dressler, R.
Duttera, W. S.
Egan, W. G.
Ellis, P. H.
Engelson, H. R.
Espenlaub, W. C.
Fine, A. M.
Fink, A.
Finn, P. L.
First, D.
Firth, L. G., Jr.
Forker, L. W.
Frank, F. P. E.
Frank, R. L.
Frankfort, M. J.
Frantz, W. P.
Freeman, H.
Friedman, E. D.
Gauger, R. H.
Gery, S. W.
Glixon, H. R.
Gneses, M. I.
Goldman, S.
Goodman, D. M.
Goodstein, J.
Gore, J. H.
Greenway, T. H.
Gretz, R. W.
Gross, S. H.
Hall, R. L.
Hansel, P. G.
Harvey, T.
Hauser, A. A., Jr.
Haynes, N. M.
Hershler, A.
Hinchey, R. L.
Horvath, W. J.
Hutchinson, T. C.
Immarco, A. E.
Jackson, D. E.
Jahn, D. M.
Johl, M. J.
Kahn, A.
Kahn, L. R.
Kassler, R.
Kawaller, J.
Kiessling, C. E.
Klein, M. M.
Kokkinos, C.
Kornbluth, A. E.
Kuhn, K. H.
Landauer, W. E.
Laspina, C. A.
Lebenbaum, M. T.
Leck, D. E.
Levenstein, H.
Lewinstein, M.
Loughlin, B. D.
Lowenschuss, O.
Lowman, R. V.
Lustig, H. E.
Mancke, R. B.
Marston, R. S.
McPhee, A. J.
Metzner, J. J.
Mosolgo, R. E.
Murphy, R. B.
Nelson, R. T.
Niewenhous, J. H.
Okwit, S.
Orazio, A. F.
Org, R. D.
Osherowitz, J. M.
Packard, K. S., Jr.
Packer, L.
Palmer, W.
Peterson, H. O.
Potter, J. T.
Poza, F.
Prew, H. E.
Prihar, Z.
Rehberg, C. F.
Reitlinger, A.
Relis, M. J.
Rich, M. W.
Richman, D.
Richter, F. G.
Rosen, A.
Rubin, W. L.
Ruvin, A. E.
Schimmel, H.
Schmerler, S.
Schneider, R. E.
Schreitmueller, R. F.
Schwartz, L. S.
Shanahan, W. J.
Shane, D. C.
Sherry, L. I.
Shivack, I. M.
Shostak, M.
Shutterly, H. B.
Silverstein, I.
Skwarek, F. J.
Smith, H. H.
Spann, R. G.
Spatz, S.
Stark, E. W.
Stateman, M. J.
Stern, E.
Stern, K.
Sussman, N.
Taylor, J. M.
Turner, E.
Urbanik, J. G.
Vadus, J. R.
Vogel, E.
Wang, C. C.
Weston, S.
Wheeler, H. A.
Wilson, L. B.
Winzemer, A. M.
Young, V. J.
Zadoff, S. A.
Zellers, J. E., Jr.

New York

Albanese, A. P.
Amarel, S.
Amatniek, E.
Andres, R. K.
Argeullo, R. J.
Armstrong, R. M.
Armstrong, R. W.
Athey, S. W.
Attwood, S. W.
Aubrey, G.
Avakian, E. A.
Bambara, J. E.
Banner, L.
Barker, D. R.
Barrios, A. A.
Baum, L.
Baum, M. C.
Baum, R. F.
Baumann, R. H.
Beach, R.
Bergen, A. R.
Berger, F. B.
Bergstein, L. E.
Bernstein, B.
Bernstein, M. H.
Bernstein, R. I.
Bertram, J. E.
Bibbero, R. J.
Bickel, H. J.
Bloom, F. J.
Bogar, J. E.
Bomberger, D. C.
Bramhall, F. B.
Brandt, W. E.
Brookner, E.
Brooks, F. P., Jr.
Brown, F. L.
Brown, P. M.
Brown, R. D.
Bruns, W. H.
Burkhard, H. F.
Byam, L. A., Jr.
Cassidy, J. J.
Cerier, M. M.
Chang, S. S. L.
Chanzit, L.
Chen, T. S.
Christian, J. L.
Claassen, F. S.
Clarke, K. K.
Clemens, G. J.
Cloud, P. R.
Cooper, F. S.
Cooper, P. W.
Cotter, M. A.
Coughlin, J. J.
Cravis, H.
Credle, A. B.
Cressey, J. J. J.
Cuccia, J. F.
Cypser, R. J.
Deblasio, C.
Demuth, H. B.
Deschamps, G. A.
Deutsch, S.
Dolobowsky, I.
Doody, D. T.
Drubin, M.
Dunford, W.
Eisenberg, L.
Ellsworth, J. L.
Epstein, M.
Exselsen, R. M.
Fagin, S. L.
Feldman, E. F.
Fifer, S.
Fisher, G. J.
Fleisher, H.
Fontaine, A. B.
Fox, J.
Freed, A.
Freiman, C. V.
Freudenberg, B.
Friedland, B.
Frum, A.
Gaffney, J. E., Jr.
Gardner, R. K.
Garman, R. L.
Gatzke, F. W.
Geduldig, D. S.
Geldston, S.
Gersch, W. M.
Gilman, G. W.
Gipslis, P. W.
Glantz, L. M.
Golay, M. J. E.
Gold, P. J.
Goldberg, B.
Golden, D.
Golden, R. K.
Goldfischer, L. I.
Goldmuntz, L. A.
Goldsmith, A. N.
Goodall, W. M.
Goronin, A.
Greibach, E. H.
Greig, J. H.
Griffith, J. E.
Groginsky, H. L.
Gronner, A.
Gruenberg, E. L.
Guenther, R.
Gusman, L.
Haanstra, J. W.
Haibt, L. H.
Hamer, H.
Hare, V. M., Jr.
Harris, B.
Harris, L.
Hauptschein, A.
Hernstadt, W. H.
Hipple, J. A.
Hogan, J. V. L.
Honig, W. M.
Jacobs, A. B.
Jaworski, A. M.
Jensen, R. A.
Julie, L.
Kahn, W. K.
Kapolwitz, M.
Karson, A.
Kassel, A.
Keller, E. W.
Kenney, D. P.
Kim, W. H.
Klein, J. A.
Koff, R. M.
Konig, W. F.
Kopel, P. S.
Krauer, O.
Laemmel, A. E.
Lampert, L.
Lapatine, S.
Lavi, Y.
Lazarus, J. P.
Lennon, T. J.
Lesman, E.
Levin, M. J.
Levy, C.
Liebman, P. M.
Lindner, H. G.
Logue, J. C.
Love, G. V.
Low, F.
Luiggi, R. H.
Lynch, F. T.
Mack, A.
Maedel, G. F.
Maricinkowski, H. L.
Marcy, H. T.
Markham, G. W.
Marshall, S. L.
McMillan, B.
Mertz, P.
Meslener, G. J.
Mico, G.
Milazzo, S.
Miller, K. S.
Miller, S. E.
Mitchell, D.
Morton, A. R.
Mountain, J. D.
Oden, P. H.
O'Hare, R. J.
O'Neill, L. H.
Onyshkevych, L. S.
Orefice, G. T.
O'Shea, M. C., Jr.
O'Sullivan, T. J.
Parker, J. A.
Parr, A. F. W.
Perlman, S.
Piore, E. R.
Rall, H. L.
Rapaport, H.
Rappolt, F. A., Jr.
Reader, M.
Reynolds, G.
Rice, S. O.
Richardson, E.
Ripp, H.
Roberts, R. P.
Rochester, N.
Romano, M.
Rosenberg, P.
Ross, A. H.
Roston, R. J.
Roth, S. S.
Rowe, H. E.
Rubin, S.
Rugge, R. A.
Russell, W. J., Jr.
Sabto-Agami, J. C.
Sakitt, M.
Samuel, A. L.
Sarahan, B. L.
Sardo, W.
Saron, R. B.
Schetzen, M.
Schillinger, A. G.
Schlam, D.
Schwartz, M.
Schwarz, R. J.
Schwenzfeger, E. E.
Selender, H.
Shakun, M. F.
Sheldon, M.
Sherman, S.
Silverman, R. A.
Sochet, E.
Sokol, S.
Soreny, E. V.
Spencer, G. D.
Sprung, J.
Steinberg, I.
Stern, R. M.
Stiefel, R. C.
Strasser, S.
Stuckert, P. E.
Sunde, E. D.
Szekely, Z.
Tabor, J. H.
Taylor, W. H., Jr.
Temes, C. L.
Tetley, W. H.
Thomas, R. O.
Trachtenberg, A.
Trapnell, F. M., Jr.
Truxal, J. G.
Tsao, T. C.
Tuckerman, L. S.
Tuttle, C. W.
Ur, H.
Vigants, A.
Vitorovich, N.
Voelcker, H. B., Jr.
Vogt, G. F.
Wallace, A.
Wallace, M.
Walston, C. E.
Walton, J. S. V.
Wasserman, R.
Weber, E.
Weber, J. H.
Weber, L. A.
Weinstein, H. M.
Wenglin, S.
Wenner, J. W.
Whitenack, R. M.
Williams, R. J.
Wilmotte, R. M.
Wilson, I. G.
Wing, O.
Wise, R. J.
Wolensky, W.
Wolfson, R.
Wood, J. R.
Wynne, W. M.
Yuan, S.
Zadeh, L. A.
Zaslavsky, S.
Zebrowitz, A. R.
Zucker, M. S.

Northern New Jersey

Aaron, M. R.
Anderson, G. M.
Assadourian, F.
Atkin, J.
Bagno, S. M.
Bailey, R. S.
Baldwin, G. L.
Belamarich, J. P.
Bell, D. T., Jr.
Belland, E. N.
Bennett, W. R.
Black, H. S.
Boubli, E. J.
Brogan, J. M.
Brown, A. T., III
Brown, C. S.
Browne, G. W.
Burford, T. M.
Burkhart, W. H.
Busignies, H.
Cadden, W. J.
Carmichael, R. L.
Clavier, A. G.
Compton, H. B.
Crowell, M. H.
Curry, T. F.
David, E. E., Jr.
Depp, W. A.
Dern, H.
De Rosa, L. A.
Desoer, C. A.
De Vany, B. V.
Dickey, D. W.
Dietzold, R. L.
Doba, S., Jr.
Dobrowolski, R. M.
Drenick, R.
Farber, D. J.
Farrow, C. W.
Fischer, L. G.
Gardner, J. L.
Goeller, L. F., Jr.
Graham, R. E.
Grandmont, P. E.
Gregory, L. A., Jr.
Groce, J. C.
Gross, F. J.
Grower, B. B.
Hamming, R. W.
Harmon, L. D.
Hartley, R. V. L.
Heiser, W. H.
Hibbert, J. J.
Hoberman, M. J.
Howard, W. A.
Huang, R. Y.
Huber, G. H.
Humphrey, R. M.
Irvine, M. M.
Jagerman, D. L.
Jensen, A. G.
Johnson, G. E.
Jones, J. J.
Julesz, B.
Kamentsky, L. A.
Karnaugh, M.
Kimme, E. G.
King, J. H., Jr.
Kovarik, H.
Kratsch, P. H.
Kreer, J. G., Jr.
Kretzmer, E. R.
Kunkel, E. A., Jr.
Kups, E. F.
Lee, C. Y.
Lender, A.
Levinson, J.
Lewis, R. F.
Light, H. A.
Lund, N.
Lundburg, F. J.
Lupo, F. J.
Martin, J. F. P.
Martin, S. J.
Mathews, M. V.
McCluskey, E. J., Jr.
McDonald, H. S.
McLean, J. B.
Mealy, G. H.
Meier, W. L.
Mock, W. C., Jr.
Montcalm, S. R.
Moore, E. F.
Mount, E.
Mueller, P. L.
Murray, A. A.
Myers, G. H.
Noll, J. C.
Norton, E. A.
Nyquist, H.
Orihuela, A.
Packer, S. S.
Panter, P. F.
Paull, M.
Perry, A. D., Jr.

Pickholtz, R. L.
Pierce, J. R.
Press, M.
Rea, W. T.
Reed, S. C.
Richards, E. G.
Richards, G. P.
Rizzo, J. E.
Segers, R. G.
Shangraw, C. C.
Shotliff, L. A.
Slepian, D.
Spanos, W. M.
Spencer, A. E., Jr.
Sullivan, H.
Sullivan, H. J.
Szczebak, E. J.
Thomas, G. N.
Thompson, G. S.
Tongue, B. H.
Treuhaft, M. A.
Ulrich, W.
Waldhauer, F. D.
Ware, P. M., Jr.
Whaley, H. R.
Wickliffe, P. R., Jr.
Winner, R. N.
Winter, R. A.
Wintringham, W. T.
Wolfe, R. M.
Zayac, F. R.

Princeton

Della-Torre, E.
De Versterre, W. I.
Dow, O. E.
Fich, S.
Fisher, R. L.
Fryer, W. R.
Garretson, E. B.
Gross, J.
Helms, H. D.
Lindsay, F. A.
Maitra, K. K.
Martin, R. D.
Mather, N. W.
McCoy, D. S.
Meyerhoff, A. A.
Norton, J. A.
Powers, K. H.
Schorr, H. I.
Shomer, J. E.
Sklansky, J.
Surber, W. H., Jr.
Thomas, D. G.
Thomas, J. B.
Thompson, T. H.
Truitt, T. D.
Westneat, A. S., Jr.
Williams, T. R.

Region 3

Atlanta

Blakely, C. E.
Bradfute, G. A., Jr.
Clark, M. C.
Corriher, H. A., Jr.
Dasher, B. J.
Finn, D. L.
Flynt, E. R.
Gremer, C. E.
Hawthorne, G. B., Jr.
Hayes, R. D.
Meadows, H. E., Jr.
Rouffy, F. E., Jr.
Schenck, J.
Taylor, J.
Wright, W. W.

Baltimore

Bang, B. A.
Becker, J. A.
Blasbalg, H.
Broady, S. N.
Choksy, N. H.
Cichanowicz, H. J.
Csepely, J. A.
Edwards, R. L., Jr.
Esterson, G. L.
Fiege-Kollmann, H.
Filipowsky, R.
Gilbert, G. B., Jr.
Glaser, E. M.
Goetz, L. P.
Gore, W. C.
Gott, E.
Green, B. A.
Herwald, S. W.
Huddleston, F. J.
Huggins, W. H.
Jacomini, O. J.
Jones, L. G. F.
Kenny, B. C.
Knox, P. C.
Kovasznay, L. S. G.
Krakau, W. S.
La Bree, C. T.
Lamb, W. G.
Lauderdale, L. K.
Lynch, D. S.
Malloy, J. J.
Maxwell, D. J.
McAuliffe, G.
McMullen, C. G.
Meador, A. B., Jr.
Mester, J. C.
Obermayer, R. W.
Ogg, F. C., Jr.
Pincoffs, P. H.
Pipkin, J. E.
Pullen, K. A., Jr.
Raudsep, I. G.
Raynes, H. D.
Real, P.
Richard, R. H.
Rogers, C. L., Jr.
Rosenblum, I. I.
Schulkin, M.
Smith, E. L., Jr.
Stefan, R.
Thomas, J. F.
Vierling, H. D.
Visher, W. A.
Watts, H. M.
Weidner, E. J.
Worrell, E. A.
Zabawa, J. J.

Central Florida

Anderson, G. F.
Beach, A. R.
Cahill, W. J.
Chen, W. H. W.
Downs, J. W.
Friedland, M. S.
Garrott, W. L.
Hamerman, L.
Horton, W. H.
Koning, R. E.
Levengood, J. W.
Mathews, B. E.
Meyerson, M.
New, C. H.
Peterson, W. W.
Pierson, C. D., Jr.
Rhodes, D. R.
Tanguay, A. R.
Widerquist, V. R.

Florida West Coast

Bellare, D.
Dingley, E. N., Jr.
Fleischer, K. M.
Marion, J. F.
Rosenzvaig, J. E.
Wells, E. L.

Huntsville

Baker, R. A.
Hallowes, J. P., Jr.
Moravek, P. H.
Morgan, C. C., Jr.
Perry, B. A.
Pittman, W. C.
Schumann, F.
Stephens, R. A.

Miami

Bajus, J. C.
Lampkin, G. F.
Myers, S. J.
Nesbitt, W. E.
Pitman, G. R., Jr.
Rose, R. M.

North Carolina

George, R. T., Jr.
Nance, R. L.
Tillman, J. E.

Philadelphia

Alexander, F. C., Jr.
Annett, M. E.
Bachofer, H. L.
Bargellini, P. L.
Beane, T. E.
Beck, J. J.
Benham, T. A.
Bensky, L. S.
Berkowitz, R. S.
Bicking, H. P.
Booker, R. W.
Booth, A. T., Jr.
Bradford, C. E.
Bradley, W. E.
Bruchlos, H. O.
Bucher, T. T. N.
Bushnell, R. H.
Bycer, B. B.
Byers, A. C.
Carter, W. S., Jr.
Charton, S.
Chronister, W. M.
Coffey, E. W.
Cohen, I.
Cohen, P.
Colodny, S. H.
Cornell, J. A.
Courtney, J. E.
Cramer, B. G.
Crawford, C. S., Jr.
Curtis, W. C.
De La Cuesta, H.
Deutsch, J.
Deutsch, R.
Ehrich, W. G.
Elizondo, E. L.
Fabbioli, L. F.
Fenton, F. H., Jr.
Fischbeck, K. H.
Flomenhoft, M.
Foley, G. M.
Friend, A. W.
Fuchs, A. M.
Gantert, A.
Garber, H. N.
Gaynor, E.
Geselowitz, D. B.
Goblick, T. J., Jr.
Gollub, R.
Goodrick, R. H.
Gottschalk, J. M.
Gray, H. J., Jr.
Greenfield, A.
Halpern, H. M.
Harris, W. A.
Hartnett, E. J.
Hellerman, H.
Herscher, M. B.
Hitt, J. J.
Hoger, D. T.
Holshouser, J. R., Jr.
Honda, H.
Howery, R. W.
Hurford, W. L.
Ingerman, P. Z.
Joseph, D.
Joshi, A. K.
Kanal, L. N.
Kashtan, S.
Katz, E. S.
Koos, E. E.
Kozikowski, J.
Krendel, E. S.
Ku, Y. H.
Lazinski, R. H.
Lisicky, A. J.
Liston, J.
Lockhart, J. C.
Lynch, C. V., Jr.
Mah, L.
Maron, I.
Mathes, R. E.
Mauchly, J. W.
McCollor, R. L.
McCracken, L. G., Jr.
McKinney, R. S.
Mertens, L. E.
Milewski, C. A.
Mittelman, W.
Newman, E. L.
Noel, R. Y. M.
Osbahr, B. F.
Patterson, G. W.
Potosky, M.
Preston, G. W.
Price, R.
Randall, N. C.
Regan, J. V., Jr.
Ringer, H. N.
Roberts, F. R.
Roberts, M. J.
Rogers, R. F.
Rojas, R. R.
Roop, R. W.
Rosen, G.
Rosenzweig, G. E.
Rothwarf, F.
Ryan, V. F., Jr.
Safren, H. G.
Saloon, J. A.
Scholtes, P. W.
Shennoy, R. P.
Sherman, S. M.
Showers, R. M.
Shucker, S.
Smith, B. J.
Sohon, H.
Sorkin, C. S.
Stauffer, R. E.
Steele, R. W.
Steinberg, B. D.
Sublette, I. H.
Taenzer, E.
Taylor, J. R.
Tompkins, H. E.
Tweet, B. O., Jr.
Urkowitz, H.
Van Gelder, A.
Walsh, D. E.
Weiner, J. R.
Weinger, R.
Weisbecker, J. A.
Weiss, E.
Wilcox, R. C.
Williams, A. J., Jr.
Woerner, L. G., Jr.
Wolin, S.
Woll, H. J.
Wong, S. Y.
Yamada, H.
Yang, T.
Zebrowitz, S.

Virginia

Conable, J. H.
Corpening, A.
Dial, E. W.
Fossum, T. T.
Georgallis, G. C.
Harris, O. R.
Harvey, G. L.
Hastings, C. E.
Maxwell, M. S.
Morton, R. W.
Noble, L. V.
Premo, D. A.
Taylor, M. L.
Welch, A. A.

Washington, D. C.

Alderson, W. S.
Alexander, H. R.
Alexander, S. N.
Allen, D. A.
Anders, F. W.
Ballard, A. H.
Bareau, A. R.
Bauer, P. S.
Baxter, J. L.
Bernstein, B.
Blackburn, C. A.
Blair, C. R.
Boggs, G. E.
Brennan, L. E.
Bullock-Webster, M.
Bush, G. B.
Campanella, S. J.
Cleckner, D. C.
Coe, G. J.
Collins, J.
Conley, J. W.
Cook, F. W.
Courtis, R. P.
Cowan, B.
Diels, J. C.
Elbourn, R. D.
Ellis, B. J.
Fine, H.
Finney, W. J.
Fisher, R. H.
Fleming, J. J.
Fluhr, F. R.
Flynn, J. E.
Freeman, J. J.
Gale, M.
Gaze, R. H.
George, S. F.
Gleason, R. F.
Godsey, W. J.
Goldberg, H.
Grisamore, N. T.
Halev, I.
Haydon, G. W.
Headrick, J. M.
Hedge, L. B.
Hedrich, A. L.
Heilprin, L. B.
Herson, J. L.
Hill, A. G.
Hoben, J. C.
Hocking, L. J.
Hoffmann, G. E.
Hogan, D. L.
James, G. E.
James, W. G.
Katzin, M.
King, A. M.
King, W. P.
Kirsch, R. A.
Kirshner, J. M.
Klein, M. H.
Kohler, H. W.
Kuck, J. H.
Kullback, S.
Laine, R. O.
La Pointe, J. C.
Leiner, A. L.
Levy, J. E.
Lieberman, G.
Loda, C. J.
Lun, M. J.
Mallin, J. A.
Mauldin, H. W., Jr.
McClurg, G. H.
McGinnis, C. E.
McLaughlin, D. J.
McLeod, J. S.
Melton, B. S.
Metzger, W.
Mitchell, G. J.
Morscher, L. N., Jr.
Morse, M. S.
Neumann, A. J.
Norton, M. H.
Notz, W. A.
Ould, R. S.
Paden, D. R.
Page, R. M.
Peterson, H. L.
Petrie, G. W., III
Petritz, R. L.
Peyton, P. B., Jr.
Phillips, M. L.
Pitsenberger, J. W.
Poor, V. D.
Reagen, E. J.
Reed, S. F.
Regardh, C. B.
Reiser, D.
Robel, R. B.
Rochelle, R. W.
Rotkin, I.
Runyan, R. E.
Safford, L. F.
Schwartz, R. J.
Scott, R. M.
Scott, S. R.
Sencenbaugh, D. W.
Shapiro, G.
Sharp, N. S.
Shepard, D. H.
Singer, C. H.
Smith, B. D., Jr.
Smith, E. D.
Spetner, L. M.
Stitch, B. D.
Summers, C. R.
Swierczek, W. L.
Swramik, J., Jr.
Thompson, R. T., Jr.
Tozzi, L. M.
Tregidga, A. C.
Uglow, K. M., Jr.
Van Lunen, R. D.
Wagner, F., Jr.
Wald, B.
Waldschmitt, J. A.
Walsh, J. P.
Weihe, W. K.
Wilcox, R. H.
Willard, J. M.
Winkler, S.
Worne, B. E.
Yost, W. E., Jr.
Youden, W. W.
Zirm, R. R.

Region 4

Akron

Brown, L. C.
Burke, J. T.
Carpenter, C. P.
Diamantides, N. D.
Fordham, C. E.
Horowitz, M.
Huffman, D. L.
Kelly, C. M.
Kierstead, F. H., Jr.
Kult, M. L.
Miller, J. H.
Pressel, P. I.
Steigerwalt, O. I.

Central Pennsylvania

Baker, W. L.
Bowhill, S. A.
Harvey, H. B.
Higdon, R. V.
Key, C. L., Jr.
Knausenberger, G. E.
Lawther, J. M.
Lemley, L. W.
Lopez, A. F.
Miller, N. B.
Norris, R. S.
Stein, S.

Cincinnati

Berg, D. F.
Doerr, W. H.
Kaufman, B. A.
Meuleman, R.
Nistico, F.
Patton, T. N., Jr.
Takahashi, E. M.

Cleveland

Baerwald, H. G.
Banshak, W. G.
Collin, R. E.
Edling, E. A.
Hammill, C. W.
Kres, A. J.
Lewis, A. M.
Saltzer, C.
Tame, J. S.
West, S. S.

Columbus

Albright, R. E.
Chope, H. R.
Conlon, R. J.
Dawirs, H. N.
Harrison, R. J.
Kouyoumjian, R. G.
Travis, E. W.
Truxall, F. W.
Ward, R. C.
Warren, C. E.
Weimer, F. C.

Dayton

Agins, B. R.
Bordewisch, J. F.

Brown, G. T., Jr.
Couch, P. W.
Drouilhet, P. R., Jr.
Goldman, C. C.
Hallman, L. B., Jr.
Hooks, L. E.
Lloyd, L. H.
Mayer, J. M.
Mims, J. H.
Piety, E. W.
Simopoulos, N. T.
Stimmel, R. G.
Thompson, C. W. N.

Detroit

Barcus, R.
Bartman, F. L.
Beutler, F. J.
Blythe, R.
Book, E.
Browder, J. E.
Chuang, K.
Chute, G. M.
Craig, J. P.
Cutrona, L. J.
Doll, R. E.
Farris, H. W.
Friedberg, I. S.
Garner, H. L.
Gilbert, E. G.
Gilbert, E. O.
Harmon, S. T.
Hart, D. E.
Hazeltine, B.
Hok, G.
Johnson, E. C., Jr.
Kaufman, L.
Kazda, L. F.
Keeney, M. G.
Kilmer, W. L.
Klem, R. F.
Lindahl, C. E.
Lindsay, W. J.
Lippmann, S. A.
Macnee, A. B.
McGlinn, E. J.
McNabb, J. W.
McPherson, R. R.
Morgan, B. S., Jr.
Morita, Y.
Nakagawa, N.
O'Neal, R. D.
Otterman, J.
Peterson, G. E.
Piper, C. A.
Porcello, L. J.
Rauch, L. L.
Reiher, H. F.
Ristenbatt, M. P.
Robinson, G. H.
Scott, N. R.
Steinmann, W. L.
Talaski, C. E.
Tieman, C. R.
Tieng, K. Y.
Tokad, Y.
Walker, E. L.
Wickman, C. R.
Youkstetter, F.

Pittsburgh

Caywood, W. P., Jr.
Dean, W. C.
Feigenbaum, E. A.
Forrester, A. T.
Gannon, G. F., Jr.
Golla, E. F.
Klotzbaugh, G. A.
Margolis, S. G.
Marlowe, E. W.
O'Donnell, T. J.
Schatz, E. R.
Shaffer, L. R., Jr.
Spriggs, L. A., Jr.
Sziklai, G. C.
Tonge, F. M.

Toledo

Josenhans, J. G.

Williamsport

Webb, H. E.

Region 5

Cedar Rapids

Babcock, J. H.
Babillus, J.
Kemble, T. H.
Leverington, R. D.
Lowenberg, E. C.
Mansur, G. F., Jr.
Stover, H. A.

Chicago

Arsem, A. D.
Beam, R. E.
Berry, R. F.
Bobis, J. P.
Borrowman, J. H.
Boyd, D. M., Jr.
Brauer, H. H.
Bridges, J. E.
Carlson, G. R.
Carter, R.
Chen, C. L.
Chorney, P. L.
Cohn, G. I.
Condron, W. F.
Costa, P. J.
Dawson, J. W.
Demasy, J.
Druz, W. S.
Eilers, C. G.
Ernst, E. W.
Everitt, W. L.
Franklin, C. W.
Gerlach, A. A.
Hansen, T. A.
Henebry, W. M.
Hungerford, J. C.
Hupert, J. J.
Isabeau, J. G.
Jarvis, K. W.
Jenness, R. R.
Jones, A. H.
Jones, R. W.
Kazel, S.
Kott, W. O.
Lewis, H. A.
Li, C. C.
Ma, H. J.
MacDonald, W. F.
Magnuski, H.
Mansfield, R.
Marshall, B. O., Jr.
Messinger, H. P.
Mikulski, J. J.
Mittelmann, E.
Mittra, R.
Moe, M. L.
Moon, R. J.
Morrison, P.
Mueller, F. M.
Muerle, J. L.
Mullin, A. A.
Nordyke, H. W., Jr.
Peach, L. C.
Pye, H. C.
Rabowsky, I.
Richards, H. F.
Rohr, W. E.
Roschke, E. M.
Rubinfien, D.
Ruina, J.
Sawada, F. H.
Sayles, H. L.
Schulz, R. B.
Sherrick, D. C.
Skaperdas, D. O.
Sladkey, R. J.
Soma, R.
Sommeria, M. R.
Splitt, F. G., Jr.
Stastny, G. F.
Stewart, C. H., II
Thomas, R. G.
Van Valkenburg, M. E.
Waller, E. E.
Warnke, G. F.
Wavering, A. J.
Webb, H. D.
Wei, L. Y.
Weissman, R. M.
White, E. S.
Woveris, L. J.
Yeslin, A. R.
Zakhaim, M.

Fort Wayne

Choi, G. Y. H.
Clark, J. R.
Hessler, J., Jr.

Indianapolis

Cooper, G. R.
Masnaghetti, R. K.
Schultz, F. V.
Tou, J.

Louisville

Kain, R. Y.

Milwaukee

Asmuth, J. L.
Benedict, T. R.
Chapman, D. M.
Davidson, C. H.
Gebhard, R. F.
Glass, T. J.
Keenan, J. A.
Kushner, H. J.
Limpel, E. J.
Mrazek, D. A.
Norum, V. D.
Rideout, V. C.
Scheibe, E. H.
Schlager, K. J.
Schweppe, F. C.
Stephan, R. R.
Theiss, C. M.
Tyson, H. B.
West, B. M.

Omaha-Lincoln

Bashara, N. M.
Chamberlain, I. J.
Heim, K. E.
Wycoff, K. H.

South Bend-Mishawaka

Bardell, P. H., Jr.

Twin Cities

Bratschi, R. W.
Cohen, A.
Featherstone, R. P.
Frohbach, H. F.
Grosz, W. S.
Hardenbergh, G. A.
Hawley, C. L., Jr.
Holte, J. E.
Ludwig, J. T.
Nordstrom, J. E.
Raabe, H. P.
Robertson, A. J. L.
Sanders, R. M.
Schmitt, O. H.
Schuck, O. H.
Schultz, S. W.
Thompson, R. H.
Wiedman, R. E.
Wilson, A. P.

Region 6

Dallas

Brachman, M. K.
Cawood, J. L.
Erringer, O. W., Jr.
Fain, W. W.
Fuller, W. D.
Kettler, C. L.
Leming, T. L.
Logan, J. J.
Miller, J. C.
Mitchell, W. R.
Mut, S. C.
Sanford, A. L.
Sarrafian, G. P.
Smith, C. H.
Stanton, A. N.
Strom, L. D.
Wadel, L. B.
Weedfall, W. W.
Wilhelm, E. S.
Ziemer, D. R.

Denver

Burkhard, D. G.
Cohen, R. S.
Cook, E. E.
Coombs, W. C.
Cottony, H. V.
Daniels, W. H.
Rice, R. B.
Slutz, R. J.
Stacey, D. S.
Sugar, G. R.
Wedge, T. E.
Zanboorie, M. H.

El Paso

Carbine, I. L.
Kidwell, R. P.
Pyle, C. A.

Fort Worth

Brust, M. F.
Delaney, J. S.
Dodd, J. M.
Fletcher, C. H.
Hix, E. W.
Sissom, A. W.
Tedeschi, A.

Houston

Bobbitt, J. T.
Easterling, M. F.
Jones, H. J.
Keating, L. M.
Kolb, R. H.
Pierson, A. L., III
Rust, W. M., Jr.
Schneider, W. P.
Smith, F. C., Jr.
Suttle, A. D., Jr.
Tanguy, D. R.
Tyler, W. L.
Wischmeyer, C. R.

Kansas City

Cummings, A. J.
Findley, L. D.
Halijak, C. A.
Love, B. E.
Van Der Maas, G. J.
Wilcox, J. V.

New Orleans

Cronvich, J. A.
Gordon, E.
Hanle, R. L., Jr.
Hatfield, F. A.
McLean, L. V.

Oklahoma City

Puckett, T. H.

St. Louis

Cummings, G. M.
Furfine, A. L.
Hirsch, O. C.
Hobbs, E. W.
Keiser, B. E.
Kline, R. M.
Little, G. R.
Mohrman, R. F.
Tucker, M. F.
Watts, C. A.
Zaborszky, J.

San Antonio

Anderson, W. L.
Bostick, F. X.
Duesterhoeft, W. C., Jr.
Economy, R.
Garner, W., Jr.
Hoffman, A. A. J.
King, J. D.
McQuown, A. N., Jr.
Smith, S. E.
Stone, C. S.

Shreveport

Bains, R. W.
Nuttall, E. D.

Tulsa

Cheney, J. F.
Day, C. E.
Freeman, L. R.
Piety, R. G.
Silverman, D.
Sykora, G. E.
Wolkov, D.

Wichita

Chapman, D. C.
Haskell, H. B.
Hickey, L. F.

Region 7

Alamogordo-Holloman

Benton, C. U.
Boughton, E. M.
Jaenke, M. G.
Liston, D. H.
Lynch, A. J., Jr.
Weber, M. E.

Albuquerque-Los Alamos

Basore, B. L.
Bidwell, C. H.
Connell, J. C.
Davis, L. W.
Hayre, H. S.
Hull, D. E.
Jean, F. H.
Kanneman, T. A.
Koschmann, A. H.
Lemm, R. S.
Malmberg, A. F.
Melloh, A. W.
Scharff, J. H.
Stevens, R. R., Jr.
Swain, G. R.
Whitaker, W. A.
Williams, C. S., Jr.

China Lake

Ashbrook, F. M.
Creusere, M. C.
Herman, R. W.
Poulson, W. A.
Zilmer, D. E.

Fort Huachuca

Evjen, H. M.
Lydon, D. A.
Ternow, H. G.

Los Angeles

Adomian, G.
Adrian, D. J.
Albrecht, A.
Albright, A. R. G.
Alexander, M. A.
Alpine, P. E.
Ambrose, J. R.
Anderson, M. J.
Andrews, L. A.
Antell, S.
Armer, P.
Arnold, J. R.
Aroyan, G. F.
Arsenault, W. R.
Aseltine, J. A.
Ashby, R. M.
Ashcraft, W. D.
Ausbourne, R. K.
Avrin, J. S.
Babcock, D. F.
Babin, R. S.
Balakrishnan, A.
Barnes, J. L.
Bartholomew, H. R.
Bechtold, I. C.
Beck, L.
Bedrosian, E.
Begovich, N. A.
Bell, N. W.
Bemis, P. S.
Bergman, C. W., Jr.
Bernhard, H. A.
Bible, R. E.
Bocast, D. R.
Bond, F. E.
Bonney, R. B.
Booton, R. C.
Bower, J. L.
Boyd, G. D.
Boykin, T. R., Jr.
Brady, M. E.
Brandon, E. T.
Braun, E. L.
Braverman, D. J.
Brigden, J. K.
Briggs, J. G.
Brown, C. S.
Brown, W. E., Jr.
Bucher, F. X.
Buchman, W. W.
Buland, R. N.
Burt, C. R.
Cahn, C. R.
Cain, G. H., Jr.
Campbell, R. A.
Capps, J. W.
Carlson, C. O.
Carpenter, R. L.
Chang, B.
Chow, A. V.
Christensen, A. V.
Chu, H.
Clabaugh, R. G.
Clarke, F. H., Jr.
Clary, W. T., Jr.
Cloonan, C. B.
Cobb, H. A., Jr.
Cochran, E. D.
Cox, J. A.
Crofts, G. B.
Culver, W. H.
Cunningham, G. W.
Curl, G. W.
Daniel, D. B.
Davis, F. W.
Davis, H.
Davis, J. S.
Davis, J. W.
De Lano, R. H.
Determan, J. D.
Dethlefsen, D. G.
Dibos, R. A.
Diemer, F. P.
Downes, L. C.
Dulin, J. J.
Duncan, D. B.
Du Waldt, B. J.
Edelsohn, C. R.
Edwards, R. F.
Egger, A.
Ellis, D. O.
Ellis, W.
Emnetrey, W. R.
Epstein, R. A.
Escher, P. H.
Eschner, A., Jr.
Estrin, G.
Evfimenko, A.
Fenster, S.
Fiske, J. J.
Frankel, S. P.
Fursa, A.
Garber, L. F.
Gardner, F. M.
Gates, C. R.
Gates, H. P., Jr.
Gee, L. C.
Ghose, R. N.
Gilchriest, C. E.
Glowalla, J.
Gluth, N. P.

la, A. S.
ldstick, G. H.
uld, G. P.
w, K. P.
abbe, E. M.
aves, R. E.
eenbaum, M.
iffin, E. F.
oss, W.
ossman, J. J.
rley, L. T.
dden, F. A.
ll, K. A.
mren, S. D.
nce, H. V.
nnum, A. J.
re, G. H.
usz, W.
wkins, J. K.
yes, W. T.
dges, C. P.
ebner, D. R.
ilfron, J.
imiller, R. C.
lland, J. C.
rshberger, W. D.
yliger, G. E.
rschl, L.
bbs, L. C.
dson, W. G.
lly, C. M.
lmberg, N. A.
ney, J. F.
skinson, E. A.
ward, S. L.
udson, C. A.
grassia, F. S.
ley, C. T., Jr.
cobs, J. E.
cobson, O. M.
cobson, R. E., Jr.
mes, B. B.
mieson, J. A.
erger, J. C.
rr, P. R.
tz, L.
wano, K.
ehn, D. G.
lly, D. H.
lso, J. M.
nnedy, J. D.
ing, C. F.
ing, C. G., Jr.
shi, F. H.
lein, W. J. J.
lestadt, B.
linger, A.
nepper, R. C.
nopoff, L.
omrosky, G. L.
rack, J. J.
rill, C. K.
ubert, B. R.
der, L. J.
mbert, J. D.
rson, C. C.
wrence, A. F., III
e, H. E.
es, A. B.
han, F. W.
ondes, C. T.
vine, S. E.
vinson, R. M.
vy, E. C.
ndholm, C. R.
nnes, K. W.
rens, C. S.
uie, W.
w, H.
m, V. W.
ons, L. H.
tle, A. C., Jr.
ac Intyre, R. M.
ac Kenzie, L. G.
allett, J. D.
allinckrodt, A. J.
argolis, M.
artin, D. L.
audal, I.
cCarthy, J. W.
cCord, H. L.
cCormick, G. F.
cFarlane, M. D.
cGann, L.
cGee, R. B.
cLaughlin, J. R.
cLeod, M. G.
cRuer, D. T.
cVey, B. D.
erryman, W. R.
etzner, H. E.
eyer, D. R.
iller, E. F.
iller, J. A.
iller, R. A.
oller, R. T.
olloy, C. T.
ongan, A. J.
ontijo, R. E., Jr.
orell, C. S.
oreno, C. A.
orton, W. B., Jr.
oss, J. F., Jr.

Moyer, J. W.
Muchmore, R. B.
Myers, W. A.
Nedland, E. H.
Nelson, C. S.
Nisenoff, N.
Noland, A. R.
Norsell, P. E.
Norwood, L. R.
Parker, N. F.
Patchell, J. T.
Pedersen, C. R.
Perez, A. A.
Peringer, P.
Pfeffer, I.
Philipson, L. L.
Phister, M., Jr.
Politi, E. Y.
Potter, R. W.
Pro, S.
Quinn, T. B.
Ramstedt, C. F.
Rechtin, E.
Reedy, P. H.
Reilly, M.
Roast, R. G.
Roberson, R. E.
Romig, H. G.
Ruck, H. C.
Ruiz, M. L.
Rypinski, C. A., Jr.
Saltzberg, B.
Samuelson, H. R.
Samulon, H. A.
Schalk, N.
Schindler, M.
Schreiber, W. F.
Schultz, P. R.
Scott, R. G.
Scott, W. H., Jr.
Seltzer, L. J.
Sensiper, S.
Shaplin, T., Jr.
Silva, L. M.
Simmons, J. W., III
Sizemore, L. E.
Skinner, M. R.
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Vader, R. L.
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Vreeland, R. W.
Wallner, R. A.
Whinnery, J. R.
Wood, F. B.
Yadavalli, S. V.
Ziegler, F. W.

Seattle

Beals, E. L.
Betts, A. L.
Brock, R. L.
Carlyle, J. W.
Dorwart, R. J.
Goldberg, P. A.
Harms, R. G.
Harvey, R. A.
Keller, N. E.
Linder, W. J.
Maynard, J. E.
Parr, E. D.
Porter, H., Jr.
Rosenberry, W. J.
Ruck, G. T.
Scidmore, D. L.
Strauss, B.
Swarm, H. M.
Wall, R. E., Jr.
Waterman, H. B.
Weaver, D. K., Jr.
Weiss, R. E.

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Morris, R. V.
Stagner, G. H.

Tucson

Bard, W. E.
Forcier, R. A.
Hoehn, A. J., Jr.
Lindenberg, E. C.
Perper, L.

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Moore, B. W. L.
Strychalski, A. J.

London

Dearle, R. C.
Prothero, J. W.

Montreal

Ashfield, J. C.
Aston, J. P.
Aubry, M.
Berube, J. P. G.
Birman, G.
Bonneville, S.
Caron, J. Y.
Cunha, S. H.
Gladwin, A. S.
Glegg, K. C. M.
Harper, M. J.
Johnston, R. F.
Kelsey, E. S.
Le Bel, J.
Lees, K. C.
L'Heureux, L. J.
Lortie, A. L.
Park, K. R.
Reeves, R.
Richard, G. B.
Robichaud, L. P. A.
Surtess, W. J.
Swirsky, R.
Vaillancourt, R. M.
Zames, G.

Northern Alberta

Smith, O. A.

Ottawa

Baart, J. G.
Beneteau, P. J.
Campbell, L. L.
Dunlop, D. P.
Farkas, J.
Galbraith, R. A. H.
Glinski, G.
Hatton, W. L.
Latham, R. W.
Lazecki, S.
Luedicke, E.
McLeish, C. W.
Vandervelden, C. G.
Webb, E. L. R.

Regina

McLean, D. W. P.

Toronto

Byers, H. G.
Casselman, W. G. B.
Coats, W. D.
Foster, J. H.
Hyde, G.
Jagger, C. E.
Kates, J.
Lang, G. R.
Newhall, E. E.
Oakes, D. A.
Pike, G. E.
Poole, G.
Qua, E. W.
Ratz, H. C.
Riegler, R. L.
Sinclair, G.
Vural, B.
Webster, C. G.
Yen, J. L.

Vancouver

Burgess, R. E.
Moore, A. D.
Noakes, F.
Pirart, M. A.
Robinson, R. A.

Winnipeg

Lumley, J. N. C.

Military Overseas

Belford, K. R.
Hall, W. W., Jr.
Harcar, A. R.
Harding, N. D., Jr.
Hayes, C. R.
Hopkins, C.

Iwai, K. G.
Karp, A. L.
Porter, F. A.
Quenstedt, R. E.
Skinner, C. A., Jr.
Smithey, L. D.
Snyder, G. H.
Travis, L. J., Jr.

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Pinasco, S. F.
Valentinuzzi, M. E.

Colombia

Meek, T. J., Jr.
Scott, C. B.

Egypt

Kamal, A. A.

Israel

Baneth, M.
Deutsch, M.
Mass, J.
Schoen, T. E.
Segalov, E.
Shekel, J.
Silberfarb, E.
Winter, Z.
Ziv, J.

Tokyo

Aoi, S.
Ezoe, H.
Harada, N.
Harashima, O.
Hoshika, Y.
Ibuka, M.
Iijima, K.
Imai, H.
Ishikawa, T.
Koga, I.
Komura, S.
Konomi, M.
Kuroiwa, Y.
Matsuyuki, T.
Mikuma, F.
Minozuma, F.
Miyakawa, H.
Miyakoshi, K.
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Nakagami, M.
Nishino, O.
Nishizaki, T.
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Okada, M.
Okada, T.
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Shintani, T.
Someya, I.
Suzuki, K.
Taki, Y.
Tanabe, Y.
Tanaka, Y.
Taniguchi, F.
Togino, K.
Tomono, M.
Yamazaki, T.
Yasuda, I.

Foreign Countries

Australia

Chinn, H. R.
D'Assumpcao, H. A.
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Honnor, W. W.
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Swire, B. E.

Austria

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Zemanek, H.

Belgium

Belevitch, V.
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Bermuda

Harries, J. H. O.

Ceylon

Gnanalingam, S.

Cuba

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Czechoslovakia

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Denmark

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Ecuador

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Federation of Rhodesia and Nyasaland

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Greece

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